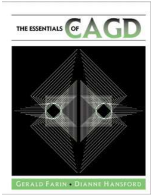
The Essentials of CAGD

Chapter 1: The Bare Basics

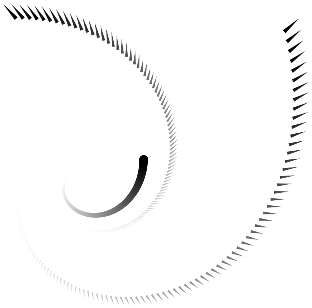
Gerald Farin & Dianne Hansford

CRC Press, Taylor & Francis Group, An A K Peters Book
www.farinhansford.com/books/essentials-cagd



Outline

Introduction to The Bare Basics



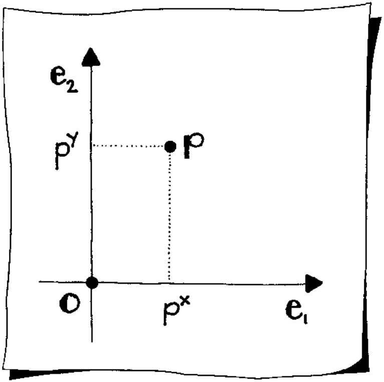
A bare basic affine mapping of a vector

Goals:

– Introduce basic geometry

– Notation

Points and Vectors

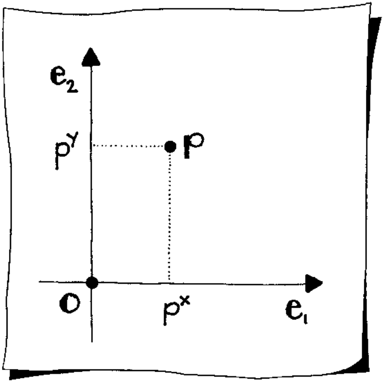


Geometry in two dimensions 2D

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For a 3D space ...

Points and Vectors



Point

– Denotes a 2D or 3D location

– Lower case boldface letters

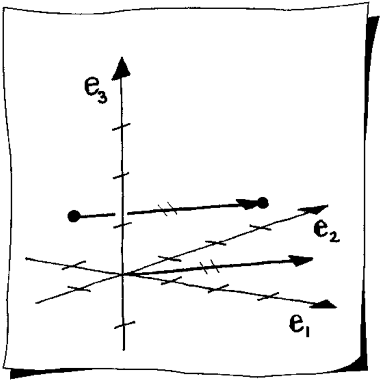
$$\mathbf{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

– Coordinates

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

Affine space or Euclidean space $\mathbb{E}^2$

Points and Vectors



Vector: difference of two points

$$\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

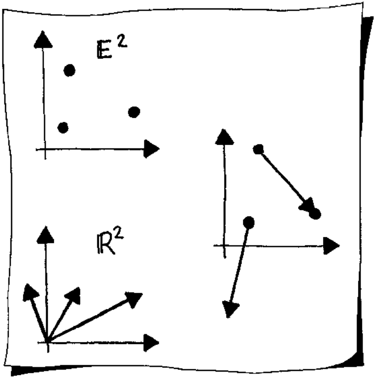
– Lower case boldface

– Components

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

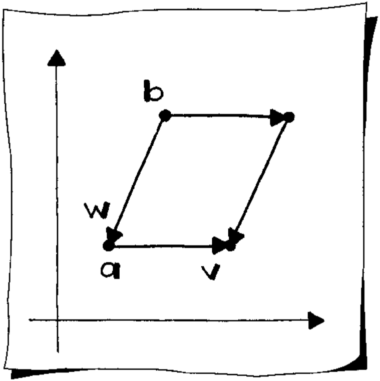
Linear space or Real space $\mathbb{R}^3$

Points and Vectors



Affine/Euclidean
and
linear/real spaces

Operations on Points and Vectors



Translation

– Moves the point by a displacement

– Displacement defined by a vector

$$\hat{\mathbf{p}} = \mathbf{p} + \mathbf{v}$$

No effect on vectors

Operations on Points and Vectors

Adding points and vectors

For vectors: Linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n, \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{R}$$

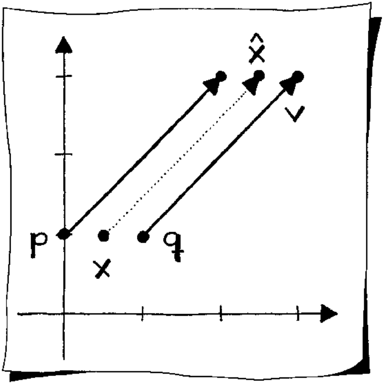
For points: barycentric combination

$$\mathbf{p} = \alpha_1 \mathbf{p}_1 + \ldots + \alpha_n \mathbf{p}_n, \qquad \alpha_1 + \ldots + \alpha_n = 1$$

What barycentric combination results in the midpoint of two points?

$$\mathbf{x} = \alpha \mathbf{p} + \beta \mathbf{q} \qquad \alpha + \beta = 1$$

Operations on Points and Vectors



Barycentric coordinates are invariant under translations

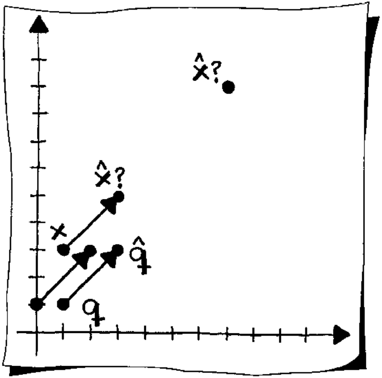
$$(\alpha \mathbf{p} + \beta \mathbf{q}) + \mathbf{v} = \alpha(\mathbf{p} + \mathbf{v}) + \beta(\mathbf{q} + \mathbf{v})$$

Sketch illustrates midpoint

$$\mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Translation vector $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Operations on Points and Vectors



The problem with non-barycentric combinations

$$\mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

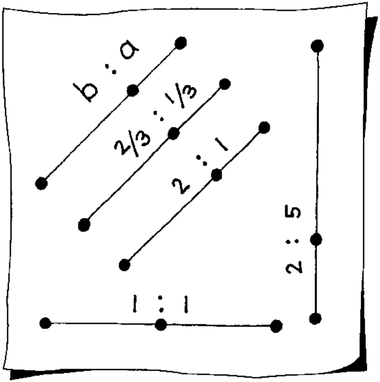
$$\mathbf{x} = 2\mathbf{p} + \mathbf{q} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Translation vector $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\hat{\mathbf{p}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \hat{\mathbf{q}} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \mathbf{x} + \mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\hat{\mathbf{x}} = 2\hat{\mathbf{p}} + \hat{\mathbf{q}} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \neq \mathbf{x} + \mathbf{v}!$$

Operations on Points and Vectors



Ratio of three (ordered) points

$$\text{ratio}(\mathbf{p}, \mathbf{x}, \mathbf{q}) = \frac{\|\mathbf{x} - \mathbf{p}\|}{\|\mathbf{q} - \mathbf{x}\|}$$

Ratios and barycentric coordinates:

$\mathbf{x} = a\mathbf{p} + b\mathbf{q}$ where $a + b = 1$

$$\text{ratio}(\mathbf{p}, \mathbf{x}, \mathbf{q}) = b : a = \frac{b}{a}$$

What if **x** not between **p** and **q**?

Products

Dot product or scalar product of vectors $\mathbf{v}$ and $\mathbf{w}$

2D: $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y$

3D: $\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$

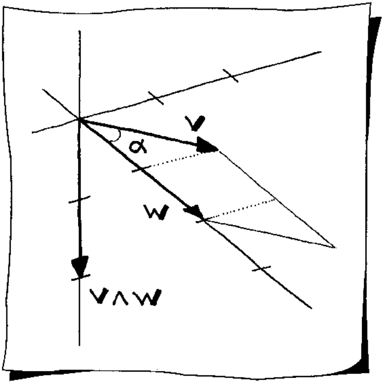
Angle α between $\mathbf{v}$ and $\mathbf{w}$:

$$\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

Length of a vector: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

When is $\mathbf{v} \cdot \mathbf{w} = 0$?

Products

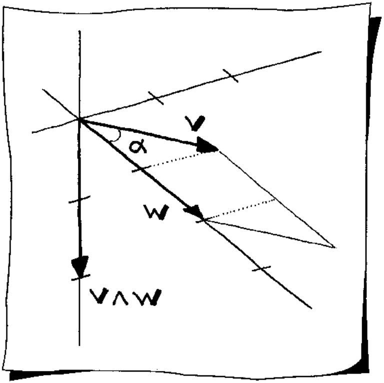


Cross product or vector product

$$\mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{bmatrix}$$

Cross product of two vectors is perpendicular to both of them

Products



Area of parallelogram spanned by **v** and **w**

$\|\mathbf{v} \wedge \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin(\alpha)$

Application: area of a triangle

When is $\mathbf{v} \wedge \mathbf{w} = 0$?

Cross products are *antisymmetric*

$$\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$$

Affine Maps

Used to move or modify a geometric figure

Given: $\mathbf{p} \in \mathbb{E}^2$ and affine map defined by 2×2 matrix A and $\mathbf{v} \in \mathbb{R}^2$

$$\hat{\mathbf{p}} = A\mathbf{p} + \mathbf{v} \quad \in \mathbb{E}^2 \quad \text{(with help of origin point)}$$

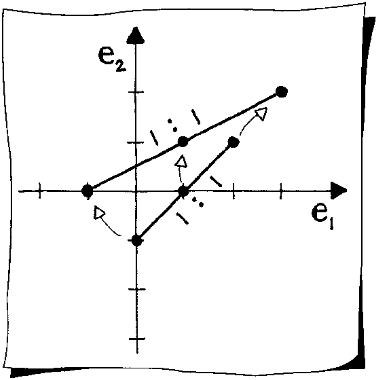
A represents a linear map

scale: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ reflection: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ projection: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

rotation: $\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ shear: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

How would you define a 3D affine map?

Affine Maps



Example

Three collinear 2D points

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Affine map

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Images of points

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Midpoint mapped to midpoint!

Affine Maps

Properties:

- Map points to points, lines to lines, and planes to planes
- Leave the ratio of three collinear points unchanged
- Parallel lines to parallel lines
  - Two parallel lines mapped to ...
  - Two non-intersecting lines mapped to ...
- Planes ...

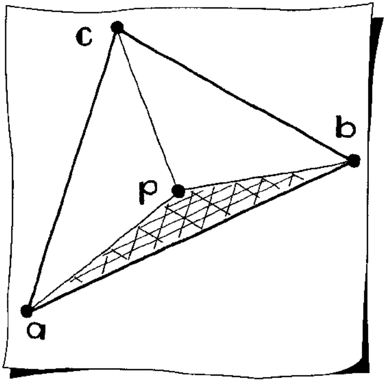
Triangles and Tetrahedra

2D triangle T formed by three noncollinear points **a**, **b**, **c**

Triangle area computed using a 3×3 determinant:

$$\text{area}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ a_x & b_x & c_x \\ a_y & b_y & c_y \end{vmatrix}$$

Triangles and Tetrahedra



Given $\mathbf{p}$ inside T

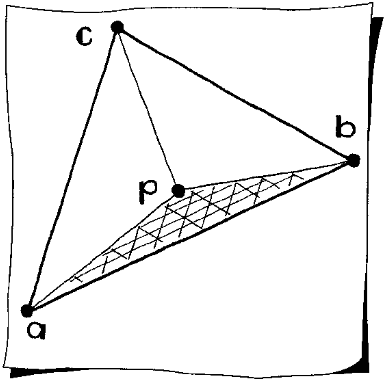
Write $\mathbf{p}$ as a combination of the triangle vertices

$$\mathbf{p} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c}$$

Combination of points
$\Rightarrow$ barycentric combination

Find u, v, w by solving
3 equations in 3 unknowns

Triangles and Tetrahedra



$$u = \frac{\text{area}(\mathbf{p}, \mathbf{b}, \mathbf{c})}{\text{area}(\mathbf{a}, \mathbf{b}, \mathbf{c})}$$

$$v = \frac{\text{area}(\mathbf{p}, \mathbf{c}, \mathbf{a})}{\text{area}(\mathbf{a}, \mathbf{b}, \mathbf{c})}$$

$$w = \frac{\text{area}(\mathbf{p}, \mathbf{a}, \mathbf{b})}{\text{area}(\mathbf{a}, \mathbf{b}, \mathbf{c})}$$

barycentric coordinates

$\mathbf{u} = (u, v, w)$

Triangles and Tetrahedra

Barycentric coordinates not independent of each other
– e.g., $w = 1 - u - v$

Behave much like "normal" coordinates:
– If $\mathbf{p}$ is given, can find $\mathbf{u}$
– If $\mathbf{u}$ is given, can find $\mathbf{p}$

Not necessary that $\mathbf{p}$ be inside T
– Need *signed* area

3 vertices of the triangle have barycentric coordinates

$$\mathbf{a} \cong (1,0,0) \qquad \mathbf{b} \cong (0,1,0) \qquad \mathbf{c} \cong (0,0,1)$$

A triangle may also be defined in 3D

$$\text{area}(\mathbf{a},\mathbf{b},\mathbf{c}) = \frac{1}{2}\|[\mathbf{b}-\mathbf{a}] \wedge [\mathbf{c}-\mathbf{a}]\|$$

Triangles and Tetrahedra

Example:

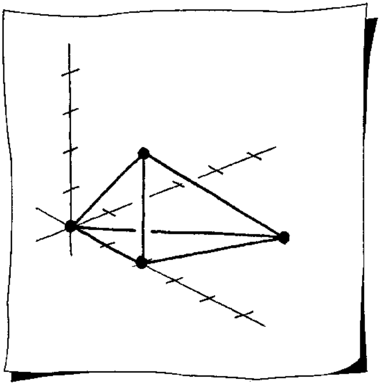
$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\mathbf{b} - \mathbf{a} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \quad \mathbf{c} - \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$\mathbf{v} = (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) = \begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix}$$

$$\mathrm{area}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{\sqrt{69}}{2}$$

Triangles and Tetrahedra



Tetrahedron: four 3D points
$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$

$$\text{vol}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \end{vmatrix}$$

Example:

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \quad \mathbf{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{vol} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 2$$