The Essentials of CAGD

Chapter 2: Lines and Planes

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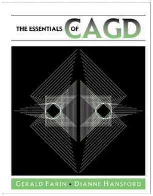
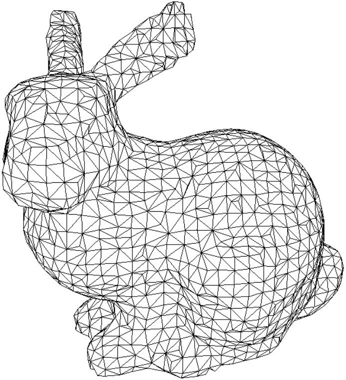
CRC Press, Taylor & Francis Group, An A K Peters Book
www.farinhansford.com/books/essentials-cagd



Outline

Introduction to Lines and Planes



Triangulation of the
"Stanford bunny"

Building blocks of
polygons and triangles

Fundamental operation:
Linear interpolation

Linear Interpolation

Points $\mathbf{p}$ and $\mathbf{q}$ define a line
How can we describe all points on this line?

Imagine a particle $\mathbf{x}$ traversing the line

Where is $\mathbf{x}(t)$ at any given time t?

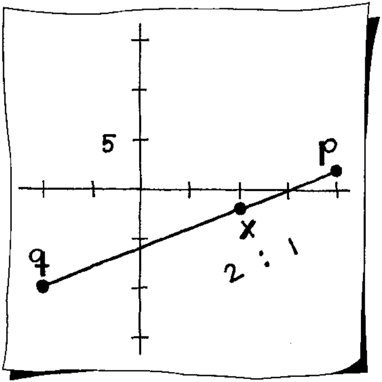
Want $\mathbf{x}(0) = \mathbf{p}$ and $\mathbf{x}(1) = \mathbf{q}$

$$\mathbf{x}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$$

This is the parametric form of a line with parameter t

What is t corresponds to the midpoint of the line segment?

Linear Interpolation



Example

$$\mathbf{p} = \begin{bmatrix} 20 \\ 2 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} -10 \\ -10 \end{bmatrix}$$

At $t = \frac{1}{3}$:

$$\mathbf{x}(\frac{1}{3}) = \frac{2}{3}\mathbf{p} + \frac{1}{3}\mathbf{q} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

Domain is real numbers

Range of the map is $\mathbf{x}(t)$

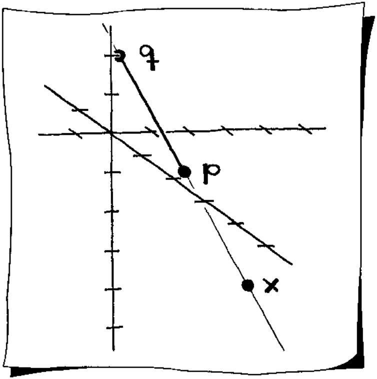
Also: preimage, image

Linear Interpolation

Linear interpolation is an *affine map*
– Ratios preserved

$$\text{ratio}(0, t, 1) = \frac{t}{1-t} \qquad \text{and} \qquad \text{ratio}(\mathbf{p}, \mathbf{x}(t), \mathbf{q}) = \frac{t}{1-t}$$

Linear Interpolation



3D parametric line

Parameter t not restricted to $[0, 1]$

Example

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

At $t = -1$:

$$\mathbf{x}(-1) = 2\mathbf{p} - \mathbf{q} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

Linear Interpolation

Line segment between $\mathbf{p}$ and $\mathbf{q}$ over parameter interval $[a, b]$

Parameter transformation: affine map taking $u \in [a, b]$ to $t \in [0, 1]$

$$t = \frac{u-a}{b-a} \qquad \text{and} \qquad 1 - t = \frac{b-u}{b-a}$$

Global parameter u and Local parameter t

Linear Interpolation

Example

$$\mathbf{p} = \begin{bmatrix} 1900 \\ 1\text{K} \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 2000 \\ 100\text{K} \end{bmatrix}$$

What is the data point for the year 1990?
The year 1990 is a global parameter
The local parameter for the line through **p** and **q**:

$$t = \frac{1990 - 1900}{2000 - 1900} = \frac{9}{10}$$

The data point for 1990:

$$\mathbf{x}(\frac{9}{10}) = \frac{1}{10}\mathbf{p} + \frac{9}{10}\mathbf{q} = \begin{bmatrix} 1990 \\ 90100 \end{bmatrix}$$

Line Forms

Parametric form: $\mathbf{x}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$

Explicit form: $y = ax + b$ (a is the slope, b is the y-intercept)

Parametric form is more general

What is explicit form of line through points

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}?$$

No explicit 3D line form

Line Forms

Convert explicit to parametric:

– Choose any two points on the line ...

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = (1-t)\begin{bmatrix} 0 \\ b \end{bmatrix} + t\begin{bmatrix} 1 \\ a+b \end{bmatrix}$$

Rewrite parametric form: $\mathbf{x}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$

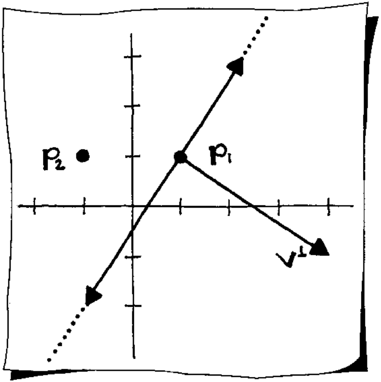
Let $\mathbf{v} = \mathbf{q} - \mathbf{p}$ and $\mathbf{v}^{\perp}$ perpendicular to $\mathbf{v}$

Point–normal form: $\quad \mathbf{v}^{\perp}[\mathbf{x} - \mathbf{p}] = 0$

Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ then point–normal form transformed to

Implicit form: $\quad ax + by = c$

Line Forms



Example

Given implicit line $3x - 2y = 1$

Are the following points on the line?

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Sketch the implicit line
using parametric form

$$\mathbf{x}(t) = \mathbf{p} + t\mathbf{v}$$

$\mathbf{p} = \mathbf{p}_1$

$\mathbf{v}^\perp = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Planes

Plane defined by three noncollinear points $\mathbf{p}, \mathbf{q}, \mathbf{r}$

Point $\mathbf{x}$ in plane:

$$\mathbf{x} = u\mathbf{p} + v\mathbf{q} + w\mathbf{r} \quad \text{where} \quad u + v + w = 1$$

Volume of tetrahedron:

$$\text{vol}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{x}) = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \mathbf{p} & \mathbf{q} & \mathbf{r} & \mathbf{x} \end{vmatrix} = 0$$

Why?

Tetrahedron with zero volume is a plane
$\Rightarrow$ $\mathbf{x}$ is in the plane spanned by $\mathbf{p}, \mathbf{q}, \mathbf{r}$

Planes

Point–vector form of a plane

$$\mathbf{x} = u\mathbf{p} + v\mathbf{q} + w\mathbf{r} = \mathbf{r} + u(\mathbf{p} - \mathbf{r}) + v(\mathbf{q} - \mathbf{r})$$

Implicit form:

Vector orthogonal (*normal*) to the plane

$$\mathbf{n} = [\mathbf{p} - \mathbf{r}] \wedge [\mathbf{q} - \mathbf{r}]$$

Point–normal form: for any point $\mathbf{x}$ in the plane

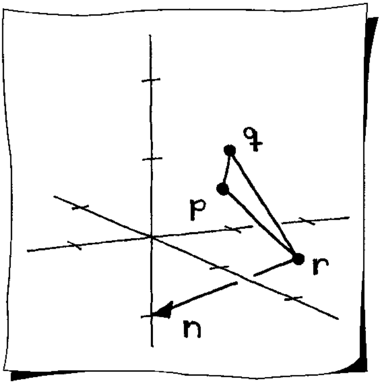
$$\mathbf{n}[\mathbf{x} - \mathbf{r}] = 0$$

Transformed to

$$ax + by + cz = d \quad \text{where} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Which form best form for {testing if, calculating} $\mathbf{x}$ in the plane?

Planes



Example

Given: three points

$$\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

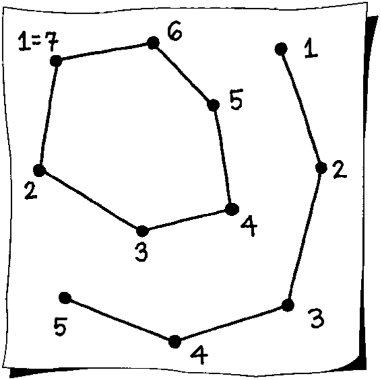
Find: implicit plane through points

$$\mathbf{n} = [\mathbf{p} - \mathbf{r}] \wedge [\mathbf{q} - \mathbf{r}] = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x - 1 \\ y - 1 \\ z \end{bmatrix} = 0$$

$$x + y + z = 2$$

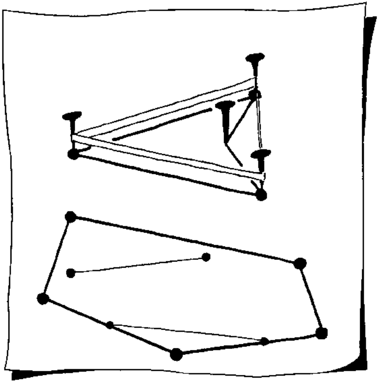
Linear Pieces: Polygons



Vertices: $\mathbf{p}_1, \ldots, \mathbf{p}_N$
Edges $\mathbf{p}_i$ to $\mathbf{p}_{i+1}$

Polygons may be *open* or *closed*

Linear Pieces: Polygons



Closed polygons classified as convex or nonconvex

Convexity tests:

1. "Rubberband" test
   Convex hull: area enclosed by rubberband
2. Line segment inclusion test

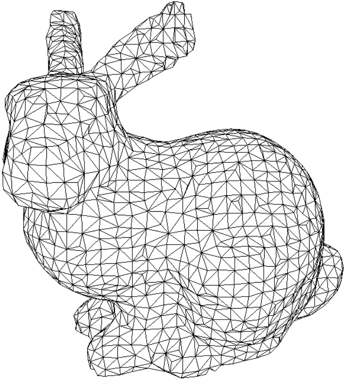
Linear Pieces: Triangulations

Most fundamental entity in computer graphics: triangles

Most rendering boils down to determining how a triangular facet interacts with the lighting model

CAD/CAM has historically used triangles as a centerpiece geometric entity for computations such as tool paths and finite element analysis (FEM)
– The reason: computations are very simple and fast

Linear Pieces: Triangulations



Triangulations can be generated through the use of laser digitizers

- Scan an object using laser rays
- Scanning generates x, y, z coordinates of points
- Software creates triangulation

Resulting triangulations tend to be large (100K + triangles)

- "Digital Michelangelo" Project (2B triangles)

Linear Pieces: Triangulations

Triangulation: a set of triangles connecting a set of points in 2D or 3D

A triangulation must satisfy the following conditions:

1. Vertices of triangles consist of given points
2. Interiors of any two triangles do not intersect
3. If two triangles not disjoint, then share a vertex or have coinciding edge
4. All triangles oriented consistently – "outward" normals

Linear Pieces: Triangulations

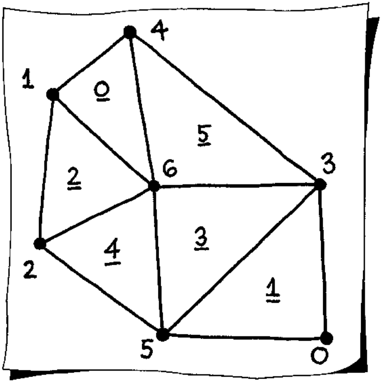
Many possibilities for a data structure – might include

- A *point list*
- A *triangle list*
- A *neighbor list*

Application dictates optimal data structure

Bad example: STL (Stereo Lithography Language)
– Data points listed multiply
– Each triangle is given explicitly by its vertices
– No neighbor information is given

Linear Pieces: Triangulations



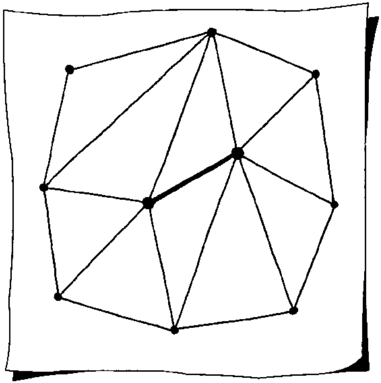
Point list:

$\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5, \mathbf{p}_6$

The triangle and neighbor lists:

triangle	vertices	neighbors
0	1, 6, 4	5, −1, 2
1	5, 0, 3,	−1, 3, −1
2	1, 2, 6	4, 0, −1
3	5, 3, 6	5, 4, 1
4	2, 5, 6	3, 2, −1
5	4, 6, 3	3, −1, 0

Working with Triangulations



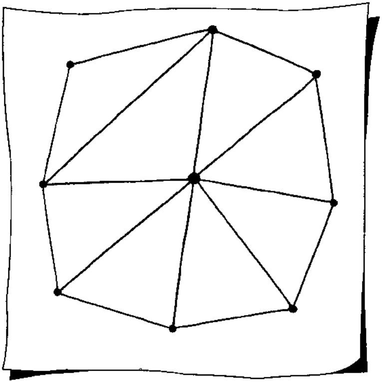
Problem: Triangulation too dense for efficient representation

Decimation:
Reduction in size of a triangulation

Remove as many triangles as possible while staying as close to the initial triangulation

Basic idea:
If multiple triangles part of one plane then replace by fewer triangles
– Planarity tolerance

Working with Triangulations



Edge Collapse Decimation:

Reduce the number of triangles by collapsing an edge

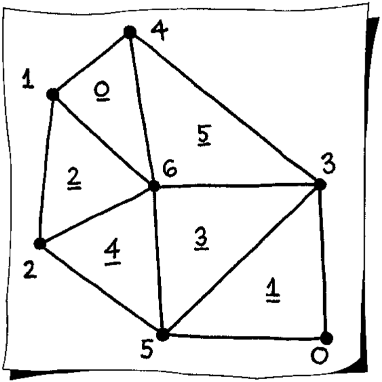
– See previous Sketch: Marked edge collapsed to midpoint

– Number of triangles reduced

– Valid triangulation maintained

– Repeat ...

Working with Triangulations



Star $\mathbf{p}^\star$ of a point $\mathbf{p}$:

Set of all triangles having $\mathbf{p}$ as a vertex

$\mathbf{p}_6^\star$ consists triangles $0, 2, 3, 4, 5$

Working with Triangulations

Edge collapse with a plane-based flatness test:

Triangulation edge $\mathbf{p}$ and $\mathbf{q}$

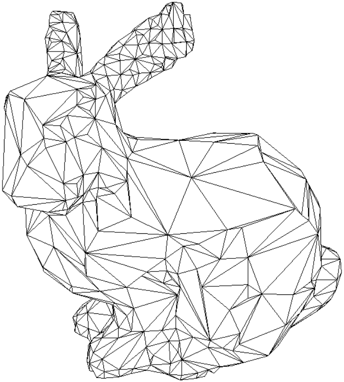
Check if all triangles formed by $\mathbf{p}^\star \cup \mathbf{q}^\star$ are sufficiently planar

– Form plane from centroid point and average normal

– All points within a given tolerance to this plane?

Yes: collapse edge

Working with Triangulations



Multiresolution representation:

Fine to coarse triangulations

Application:
transmission through the internet