The Essentials of CAGD

Chapter 3: Cubic Bézier Curves

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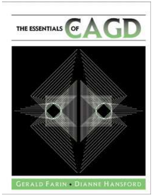
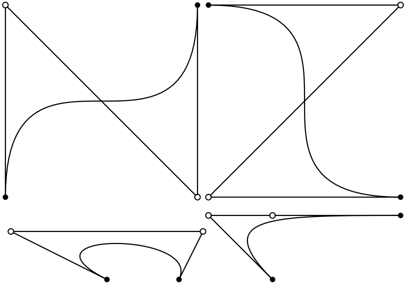
CRC Press, Taylor & Francis Group, An A K Peters Book
www.farinhansford.com/books/essentials-cagd



Outline

Introduction to Cubic Bézier Curves

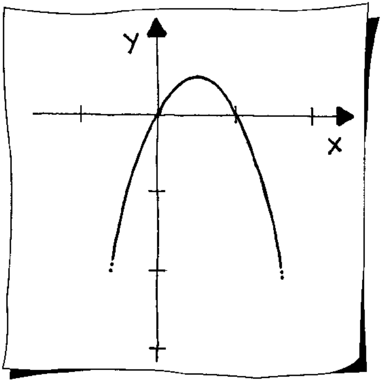


Cubic Bézier curves

– CAD/CAM
– Graphic Design
– Computer Graphics
– Figure generated in PostScript

Basic principles of Bézier curves easily explored via cubics

Parametric Curves



Curve from calculus: function

$$y = 2x - 2x^2$$

Graph of the function

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x - 2x^2 \end{bmatrix}$$

Parametric curve

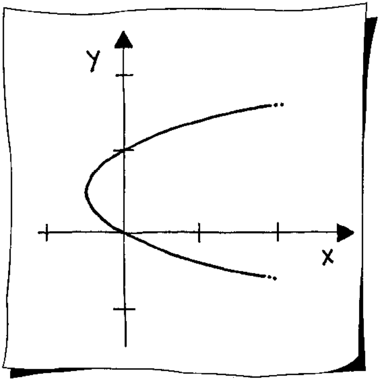
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

f and g can be any kind of function

Domain is the real line

Try this:
Parametric line thru 2 points

Parametric Curves



Graph of a function

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x - 2x^2 \end{bmatrix}$$

as a parametric curve:

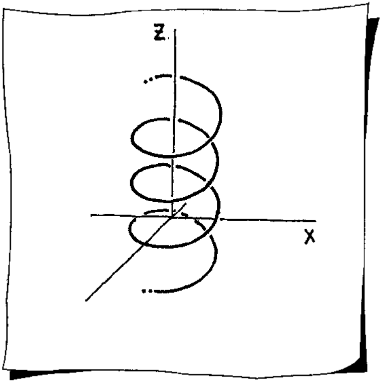
$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t - 2t^2 \end{bmatrix}$$

Rotate 90 degrees

$$\mathbf{x}(t) = \begin{bmatrix} -2t + 2t^2 \\ t \end{bmatrix}$$

Horizontal tangents do not characterize extreme points for parametric curves

Parametric Curves



Parametric curves defined in 3D:

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$$

Simple example: a *helix*

$$\mathbf{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

Cubic Bézier Curves

Focus of this book: *Bézier curves*

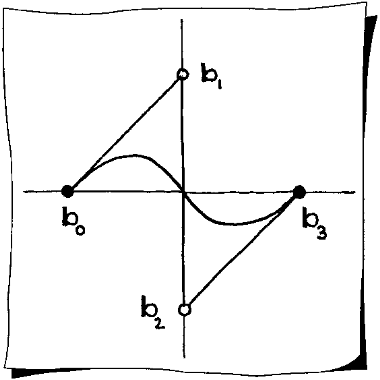
– The most important type of polynomial curve

– Named after Pierre Bézier

– Defined for any polynomial degree

– First focus on cubic case $n = 3$

Cubic Bézier Curves



$$\mathbf{x}(t) = \begin{bmatrix} -(1-t)^3 + t^3 \\ 3(1-t)^2 t - 3(1-t)t^2 \end{bmatrix}$$

Shape?

Rewrite as a combination of points

$$\mathbf{x}(t) = (1-t)^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3(1-t)^2 t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3(1-t)t^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Cubic Bézier Curves

Cubic Bézier curve

$$\mathbf{x}(t) = (1-t)^3\mathbf{b}_0 + 3(1-t)^2 t\mathbf{b}_1 + 3(1-t)t^2\mathbf{b}_2 + t^3\mathbf{b}_3$$

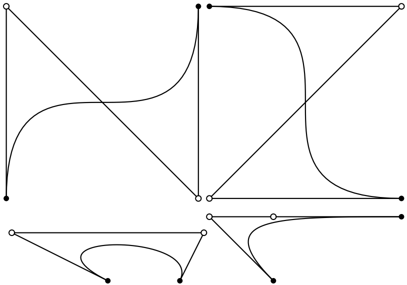
Bézier control points $\mathbf{b}_i$ form the Bézier polygon

Cubic Bernstein polynomials B_i^3

$$\mathbf{x}(t) = B_0^3(t)\mathbf{b}_0 + B_1^3(t)\mathbf{b}_1 + B_2^3(t)\mathbf{b}_2 + B_3^3(t)\mathbf{b}_3$$

More on the Bernstein polynomials in the next chapter

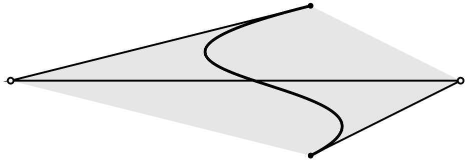
Cubic Bézier Curves



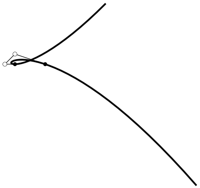
Properties:

1. Endpoint interpolation
2. Symmetry
3. Invariance under affine maps
4. Convex hull property
5. Linear precision

Cubic Bézier Curves



Convex hull property



$t \in [-1, 2]$

Extrapolation: t outside $[0, 1]$

– No convex hull property

– Unpredictable behavior

Derivatives

Tangent vector

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = -3(1-t)^2\mathbf{b}_0 + [3(1-t)^2 - 6(1-t)t]\mathbf{b}_1 + [6(1-t)t - 3t^2]\mathbf{b}_2 + 3t^2\mathbf{b}_3$$

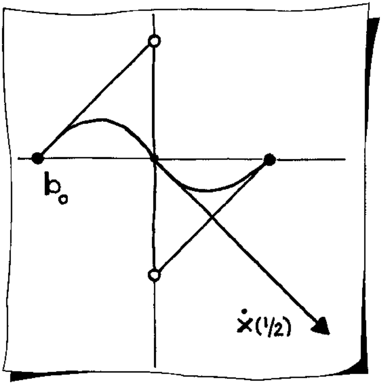
$$= 3[\mathbf{b}_1 - \mathbf{b}_0](1-t)^2 + 6[\mathbf{b}_2 - \mathbf{b}_1](1-t)t + 3[\mathbf{b}_3 - \mathbf{b}_2]t^2$$

$$= 3\Delta\mathbf{b}_0(1-t)^2 + 6\Delta\mathbf{b}_1(1-t)t + 3\Delta\mathbf{b}_2 t^2$$

forward difference $\Delta\mathbf{b}_i$

$\dot{\mathbf{x}}(t) \equiv \mathrm{d}\mathbf{x}(t)/\mathrm{d}t$

Derivatives



Example

$$\dot{\mathbf{x}}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1-t)^2 + 6 \begin{bmatrix} 0 \\ -2 \end{bmatrix} (1-t)t + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^2$$

$$\dot{\mathbf{x}}(0.5) = \begin{bmatrix} 1.5 \\ -1.5 \end{bmatrix}$$

Derivatives

$$\dot{\mathbf{x}}(t) = 3\Delta\mathbf{b}_0(1-t)^2 + 6\Delta\mathbf{b}_1(1-t)t + 3\Delta\mathbf{b}_2 t^2$$

– Derivative of a cubic curve is a quadratic curve

– Evaluating "derivative curve" produces vectors

At the curve's endpoints:

$$\dot{\mathbf{x}}(0) = 3\Delta\mathbf{b}_0 \qquad \dot{\mathbf{x}}(1) = 3\Delta\mathbf{b}_2$$

$\Rightarrow$ control polygon is tangent to the curve at the endpoints

Chapter 4: higher order derivatives

The de Casteljau Algorithm

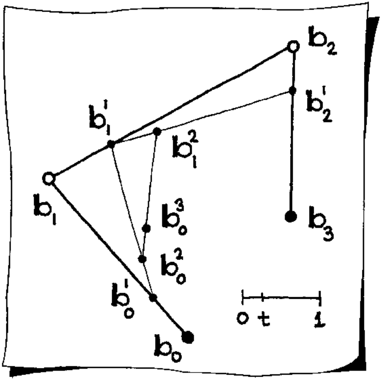
Recursive algorithm that constructs the point $\mathbf{x}(t)$ on a Bézier curve

Most important algorithm of all of CAGD

Many practical and theoretical ramifications

1959 Paul de Faget de Casteljau

The de Casteljau Algorithm



Given: $\mathbf{b}_0, \ldots, \mathbf{b}_3$ and t

Find: $\mathbf{x}(t)$

$$\mathbf{b}_0^1 = (1-t)\mathbf{b}_0 + t\mathbf{b}_1$$
$$\mathbf{b}_1^1 = (1-t)\mathbf{b}_1 + t\mathbf{b}_2$$
$$\mathbf{b}_2^1 = (1-t)\mathbf{b}_2 + t\mathbf{b}_3$$

$$\mathbf{b}_0^2 = (1-t)\mathbf{b}_0^1 + t\mathbf{b}_1^1$$
$$\mathbf{b}_1^2 = (1-t)\mathbf{b}_1^1 + t\mathbf{b}_2^1$$

$$\mathbf{x}(t) = \mathbf{b}_0^3 = (1-t)\mathbf{b}_0^2 + t\mathbf{b}_1^2$$

What operation is repeated?

The de Casteljau Algorithm

Schematic tool

$$\begin{array}{llll} \mathbf{b}_0 & & & \\ \mathbf{b}_1 & \mathbf{b}_0^1 & & \\ \mathbf{b}_2 & \mathbf{b}_1^1 & \mathbf{b}_0^2 & \\ \mathbf{b}_3 & \mathbf{b}_2^1 & \mathbf{b}_1^2 & \mathbf{b}_0^3 \end{array}$$

Implementation: 1D array data structure sufficient

The de Casteljau Algorithm

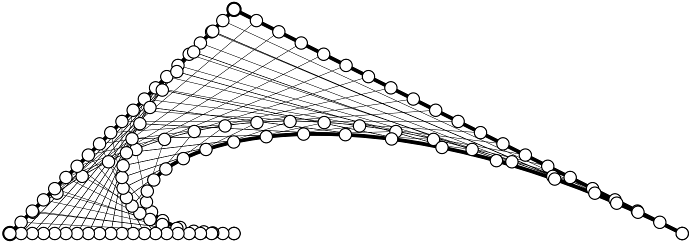
Example

$$\mathbf{x}(t) = (1-t)^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3(1-t)^2 t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3(1-t)t^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Evaluate at $t = 0.5$ using the triangular schematic tool

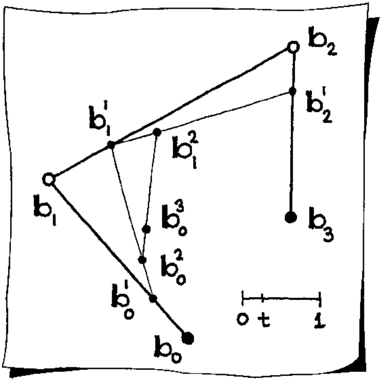
Make a sketch!

The de Casteljau Algorithm



Just for fun: All intermediate points of many evaluations

The de Casteljau Algorithm



Derivatives

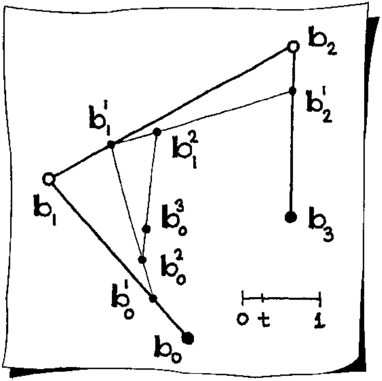
$\overline{\mathbf{b}_0^2\mathbf{b}_1^2}$ is tangent to the curve

$$\dot{\mathbf{x}}(t) = 3[\mathbf{b}_1^2 - \mathbf{b}_0^2]$$

Derivative as byproduct of point evaluation

Great value computationally

Subdivision



Curve over $[0, t]$

$$\mathbf{b}_0, \mathbf{b}_0^1, \mathbf{b}_0^2, \mathbf{b}_0^3$$

Curve over $[t, 1]$

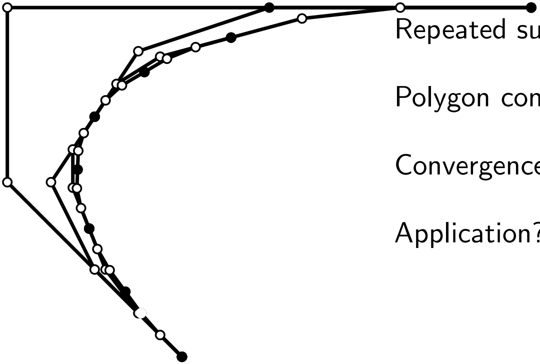
$$\mathbf{b}_0^3, \mathbf{b}_1^2, \mathbf{b}_2^1, \mathbf{b}_3$$

Find these control points:

$$\begin{array}{llll} \mathbf{b}_0 & & & \\ \mathbf{b}_1 & \mathbf{b}_0^1 & & \\ \mathbf{b}_2 & \mathbf{b}_1^1 & \mathbf{b}_0^2 & \\ \mathbf{b}_3 & \mathbf{b}_2^1 & \mathbf{b}_1^2 & \mathbf{b}_0^3 \end{array}$$

Subdivide at $t = 0.5$
Are the two curve arcs equal length?

Subdivision



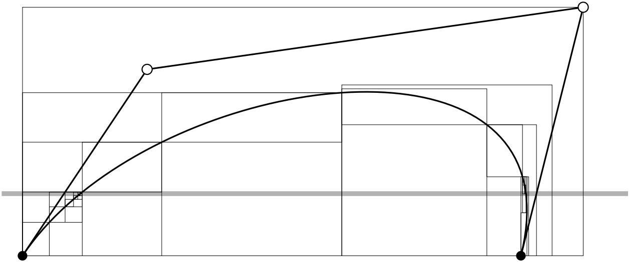
Repeated subdivision

Polygon converges to curve

Convergence is fast

Application?

Subdivision



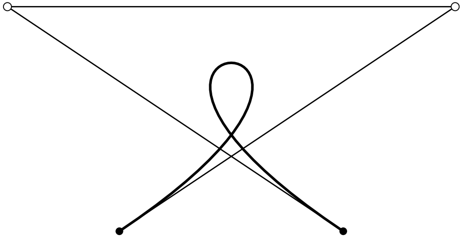
Application: curve / line intersection

True/False of minmax box / line intersection fast

How is convex hull property used?

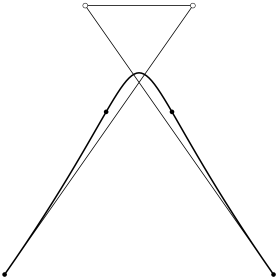
Try outlining the intersection algorithm

Exploring the Properties of Bézier Curves



Self-intersection

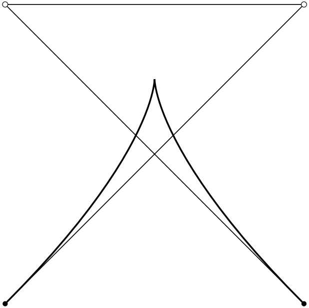
Exploring the Properties of Bézier Curves



Two inflection points

Cubic *functions* cannot have two inflection points
$\Rightarrow$ Parametric curves more flexible

Exploring the Properties of Bézier Curves



Cusp: point where the first derivative vector vanishes

See for yourself!
Run the de Casteljau algorithm for $t = 0.5$

The Matrix Form and Monomials

Cubic Bézier curve:

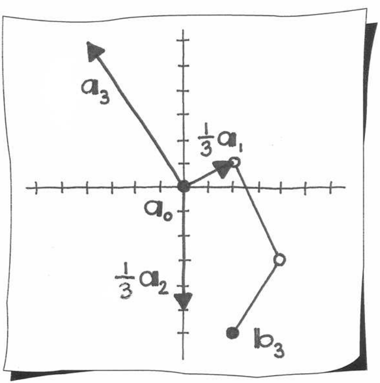
$$\mathbf{b}(t) = B_0^3(t)\mathbf{b}_0 + B_1^3(t)\mathbf{b}_1 + B_2^3(t)\mathbf{b}_2 + B_3^3(t)\mathbf{b}_3$$

Rewrite in matrix form:

$$\mathbf{b}(t) = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

The Matrix Form and Monomials



Monomial polynomials: $1, t, t^2, t^3$

Reformulate a Bézier curve

$$\begin{aligned}\mathbf{b}(t) = \mathbf{b}_0 & \\ & + 3t(\mathbf{b}_1 - \mathbf{b}_0) \\ & + 3t^2(\mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_0) \\ & + t^3(\mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0)\end{aligned}$$

$$= \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3$$

What is the *geometric interpretation* of the $\mathbf{a}_i$?

The Matrix Form and Monomials

The monomial coefficients $\mathbf{a}_i$ are defined as

$$\begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} = \begin{bmatrix}\mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} \begin{bmatrix}1 & -3 & 3 & -1\\ 0 & 3 & -6 & 3\\ 0 & 0 & 3 & -3\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Inverse process:

$$\begin{bmatrix}\mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} \begin{bmatrix}1 & -3 & 3 & -1\\ 0 & 3 & -6 & 3\\ 0 & 0 & 3 & -3\\ 0 & 0 & 0 & 1\end{bmatrix}^{-1}$$

Square matrix above is nonsingular
$\Rightarrow$ Any cubic curve can be written in Bézier or monomial form

How do we know that the matrix is nonsingular?