The Essentials of CAGD

Chapter 8: Shape

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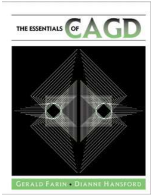
CRC Press, Taylor & Francis Group, An A K Peters Book
www.farinhansford.com/books/essentials-cagd



Outline

Introduction to Shape

Surface geometry with reflection lines

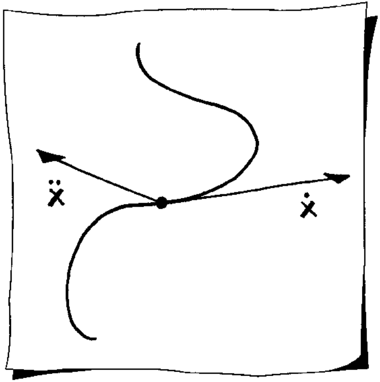
Often times a designer thinks of a curve in terms such as “fair,” “smooth,” or “sweet”

How can such concepts be incorporated into computer programs?

The central concept of any kind of shape description is *curvature*

This chapter investigates shape analysis

The Frenet Frame



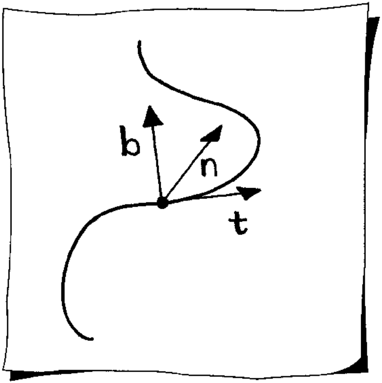
Discuss shape of a curve
in *local* terms
– Shape at a particular point $\mathbf{x}(t)$

Create a local coordinate system
at $\mathbf{x}(t)$
– Use to express local curve
properties

Base system on first and second
derivatives of the curve:

$$\dot{\mathbf{x}}(t) \quad \text{and} \quad \ddot{\mathbf{x}}(t)$$

The Frenet Frame



Local coordinate system (frame) defined by 3 vectors:
– unit length and orthogonal

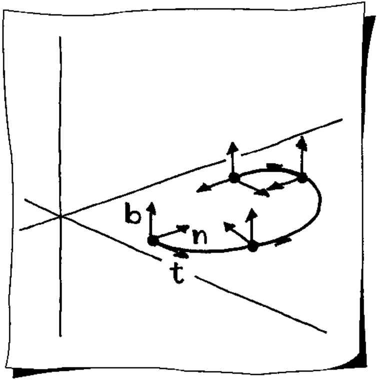
$$\mathbf{t} = \frac{\dot{\mathbf{x}}(t)}{\|\dot{\mathbf{x}}(t)\|} \qquad \text{tangent}$$

$$\mathbf{b} = \frac{\dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t)}{\|\dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t)\|} \qquad \text{binormal}$$

$$\mathbf{n} = \mathbf{b} \wedge \mathbf{t} \qquad \text{normal}$$

Called the Frenet frame at $\mathbf{x}(t)$

The Frenet Frame



Note: if either

$$\dot{\mathbf{x}}(t) = \mathbf{0} \quad \text{or} \quad \dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t) = \mathbf{0}$$

$\Rightarrow$ Frenet frame not defined

The Frenet Frame

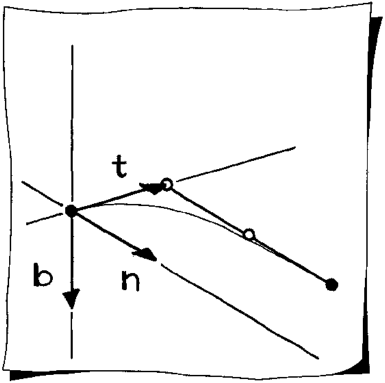
Let $\mathbf{x}(t)$ trace out points on curve

Corresponding Frenet frames also slide along the curve

Application: Positioning objects along a curve
–Letter always at the same location relative to the Frenet frame

The Frenet Frame

Example:



Curvature and Torsion

How is the Frenet frame related to the shape of a curve?

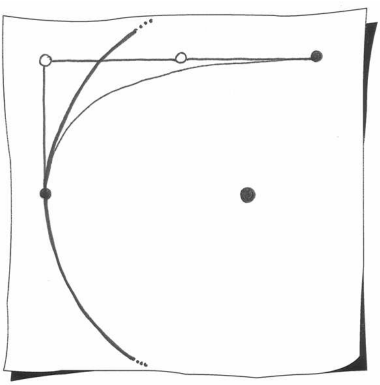
Move along the curve and observe how frame changes
– More the curve is bent $\Rightarrow$ the faster the frame will change

Rate of change of the unit tangent vector $\mathbf{t}$ denotes the curvature of the curve
– Straight line: curvature is zero
– Circle: curvature is constant

Curvature denoted by κ

$$\kappa(t) = \frac{\|\dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t)\|}{\|\dot{\mathbf{x}}(t)\|^3}$$

Curvature and Torsion



Curvature related to circle that best approximates the curve at $\mathbf{x}(t)$

– Called the osculating circle

– Radius $\rho = 1/\kappa$

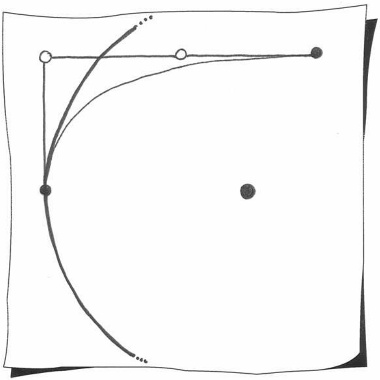
– Center

$$\mathbf{c}(t) = \mathbf{x}(t) + \rho(t)\mathbf{n}(t)$$

Osculating circle lies in the osculating plane

– Spanned by $\mathbf{t}$ and $\mathbf{n}$

Curvature and Torsion



Example:

$$\kappa(0) = \frac{2}{3} \quad \text{and} \quad \kappa(1) = 0$$

Agrees nicely with intuitive notion of "curvedness"

Center of the osculating circle at $t = 0$:

$$\mathbf{c}(0) = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$$

$\mathbf{c}(0)$ undefined since $\rho = 1/0$

Curvature and Torsion

For the special case of Bézier curves

$$\kappa(0) = 2\frac{n-1}{n}\frac{\mathrm{area}[\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2]}{\|\mathbf{b}_1 - \mathbf{b}_0\|^3}$$

$\kappa(0) = 0$ if $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ are collinear

$$\kappa(1) = 2\frac{n-1}{n}\frac{\mathrm{area}[\mathbf{b}_{n-2}, \mathbf{b}_{n-1}, \mathbf{b}_n]}{\|\mathbf{b}_n - \mathbf{b}_{n-1}\|^3}$$

Curvature at parameter values other than 0 or 1
$\Rightarrow$ Subdivide at the desired parameter value and proceed as above

Curvature and Torsion

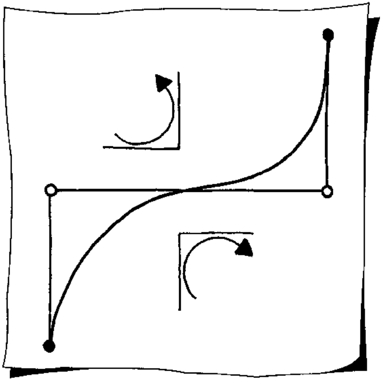
By definition a 3D curve has nonnegative curvature

For 2D curves: may assign a sign to curvature

$$\kappa(t) = \frac{\det\left[\dot{\mathbf{x}}(t) \quad \ddot{\mathbf{x}}(t)\right]}{\|\dot{\mathbf{x}}(t)\|^3}$$

Signed curvature $\Rightarrow$ inflection points
– Where the curvature changes sign
– For Bézier curves: $\mathrm{area}[\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2]$ can be assigned a sign in 2D
– Sign does not actually belong to the curvature
Indication of change in relation to the right-hand rule

Curvature and Torsion



Example:

$$\text{area}[\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2] = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} = -1$$

$$\|\mathbf{b}_1 - \mathbf{b}_0\| = 1$$

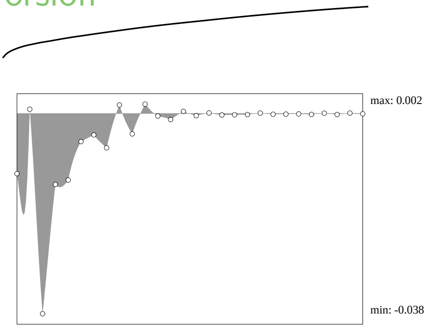
$$\kappa(0) = 2 \cdot \frac{2}{3} \cdot -\frac{1}{1} = -\frac{4}{3}$$

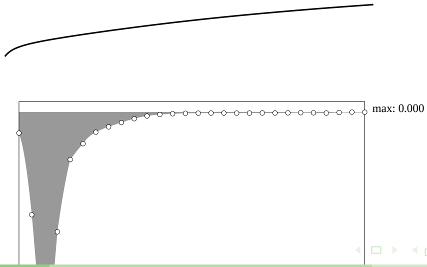
$$\kappa(1) = \frac{4}{3} \cdot \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \frac{4}{3}$$

Curvature is continuous along cubic polynomial

$\Rightarrow$ Curvature zero somewhere

Curvature and Torsion





Curvature and Torsion

The torsion τ measures the change in a curve's binormal vector

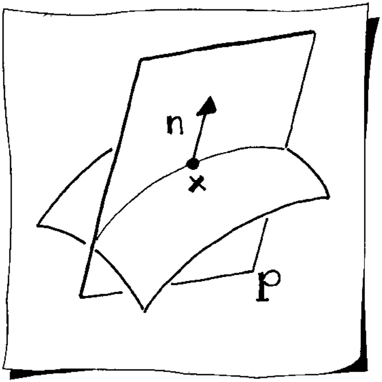
$$\tau(t) = \frac{\det[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \dddot{\mathbf{x}}]}{\|\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}\|^2}$$

The binormal of a planar curve is constant
$\Rightarrow$ a quadratic curve has zero torsion

For Bézier curves:

$$\tau(0) = \frac{3}{2} \frac{n-2}{n} \frac{\mathrm{volume}[\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]}{\mathrm{area}[\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2]^2}$$

Surface Curvatures

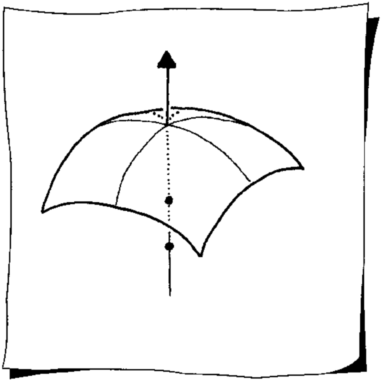


Point on surface: $\mathbf{x}(u, v)$
Normal: $\mathbf{n}(u, v)$

Plane $\mathbf{P}$ through $\mathbf{x}$ containing $\mathbf{n}$ intersects the surface in a planar curve
$\Rightarrow$ a normal section of $\mathbf{x}$

Compute signed curvature of normal section at $\mathbf{x}$
– Called normal curvature $\kappa_{\mathbf{P}}$

Surface Curvatures



Rotate **P** around **n**

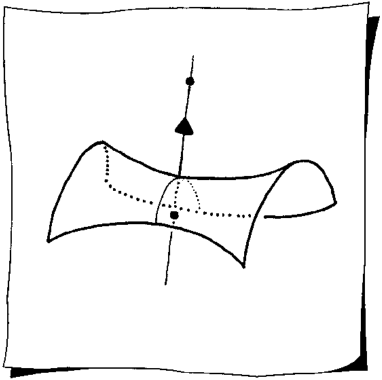
For each new position of **P**
$\Rightarrow$ New normal section
$\Rightarrow$ New normal curvature

κ_{max}: largest normal curvature
κ_{min}: smallest
$\Rightarrow$ Principal curvatures at **x**

If κ_{min} and κ_{max} both positive or both negative
$\Rightarrow$ **x** called an elliptic point
(Sphere and ellipsoid: all points elliptic)

Sketch shows center of each osculating circle

Surface Curvatures

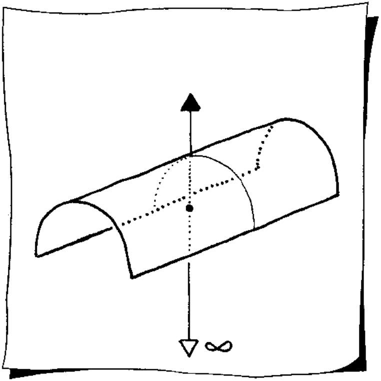


κ_{min} and κ_{max} are of opposite sign
$\Rightarrow$ **x** called a hyperbolic point
Also called saddle point

All points on hyperboloids and bilinear patches are hyperbolic

Best "real life" example of surfaces with hyperbolic points: potato chips

Surface Curvatures



κ_{min} or κ_{max} is zero
$\mathbf{x}$ is called a parabolic point

Examples: cylinders or cones

Surface Curvatures

Three cases succinctly described by

$$K = \kappa_{\text{min}} \kappa_{\text{max}}$$

Called Gaussian curvature

1. Elliptic point: $K > 0$
2. Hyperbolic point: $K < 0$
3. Parabolic point: $K = 0$

Most surfaces are not composed entirely of one type of Gaussian curvature

Developable surfaces: surfaces with $K = 0$ everywhere

Surface Curvatures

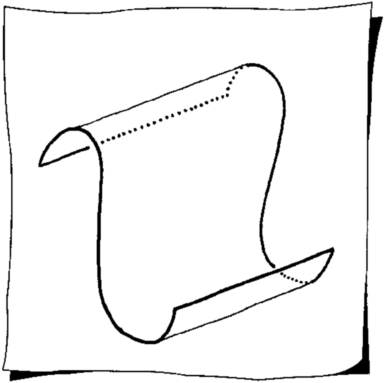
Computing Gaussian curvature:

$$F = \det \begin{bmatrix} \mathbf{x}_u \mathbf{x}_u & \mathbf{x}_u \mathbf{x}_v \\ \mathbf{x}_u \mathbf{x}_v & \mathbf{x}_v \mathbf{x}_v \end{bmatrix} \qquad \text{First fundamental form}$$

$$S = \det \begin{bmatrix} \mathbf{n}\mathbf{x}_{u,u} & \mathbf{n}\mathbf{x}_{u,v} \\ \mathbf{n}\mathbf{x}_{u,v} & \mathbf{n}\mathbf{x}_{v,v} \end{bmatrix} \qquad \text{Second fundamental form}$$

$$K = \frac{S}{F}$$

Surface Curvatures



Gaussian curvature doesn't say everything about shape

Sketch: intuitively quite curved yet $\kappa_{\min} = 0$ everywhere

Surface Curvatures

More shape measures:

Mean curvature

$$M = \frac{1}{2}[\kappa_{\text{min}} + \kappa_{\text{max}}]$$

Computed as

$$M = \frac{[\mathbf{n}\mathbf{x}_{vv}]\mathbf{x}_u^2 - 2[\mathbf{n}\mathbf{x}_{uv}][\mathbf{x}_u\mathbf{x}_v] + [\mathbf{n}\mathbf{x}_{uu}]\mathbf{x}_v^2}{F}$$

Minimal surfaces: mean curvature zero
– Such surfaces resemble the shape of soap bubbles

Absolute curvature

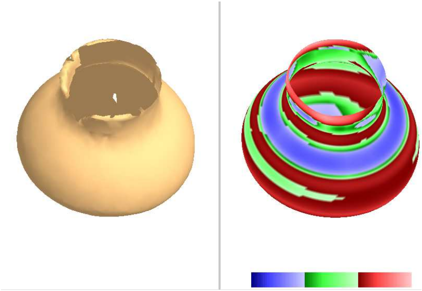
$$A = |\kappa_{\text{min}}| + |\kappa_{\text{max}}|$$

Measures the curvature of a surface in the most reliable way from an *intuitive* viewpoint

Surface Curvatures

RMS (root mean square) curvature

$$R = \sqrt{\kappa_{\text{min}}^2 + \kappa_{\text{max}}^2} = R = \sqrt{4M^2 - 2K}$$



Left: a digitized vessel Right: RMS curvatures of a B-spline approximation

Reflection Lines

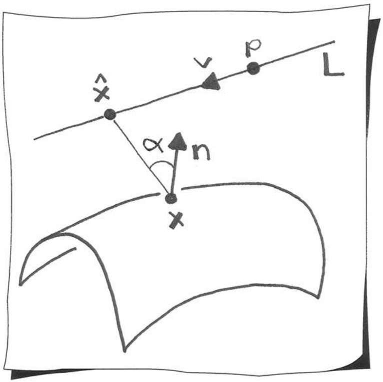
Surface curvatures – Gaussian, Mean, Absolute, RMS
Not necessarily intuitive to designers trying to create "beautiful" shapes

A different surface tool is used more often
Based on the simulation of an automotive design studio
– Car prototype built
– Placed in studio with ceiling filled with parallel fluorescent light bulbs
– Bulb reflections in car's surface
– Give designers crucial feedback on the quality of product
– "Flowing" reflection patterns are good, "wiggly" ones are bad

These light patterns can be *simulated* before a prototype is built
– Saves money: building prototype is expensive

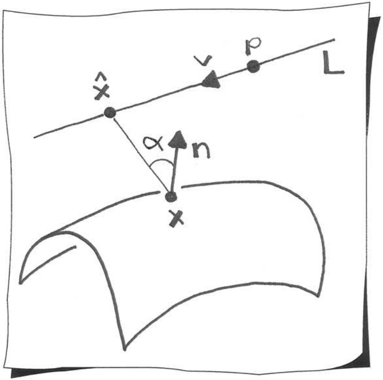
Reflection Lines

Highlight areas on surface where reflections will occur



Simple model: For any point **x**
– Compute its normal **n**
– Let **L** denote a line light source
– If α between **n** and **L** is small
normal points to light source **L**
$\Rightarrow$ Region of the surface highlighted

Reflection Lines



Compute α:
Find the point $\hat{\mathbf{x}}$ on $\mathbf{L}$ closest to $\mathbf{x}$

$\mathbf{L}$ defined by point $\mathbf{p}$ and vector $\mathbf{v}$

$$\hat{\mathbf{x}} = \mathbf{p} + \frac{\mathbf{v}[\mathbf{x} - \mathbf{p}]}{\|\mathbf{v}\|^2}\mathbf{v}$$

α given by the angle between $\mathbf{n}$ and $\hat{\mathbf{x}} - \mathbf{x}$

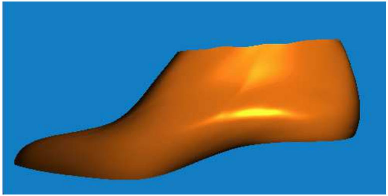
Isophote: Curve on surface determined by the light line $\mathbf{L}$

Reflection Lines

Left: B-spline surface has some shape imperfections

– Not too obvious from the shaded image

Middle: Reflection line display reveals them clearly



Smoothing algorithm applied to B-spline surface

⇒ Right: Improved reflection pattern