Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 10: Affine Maps in 3D

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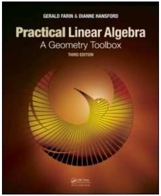
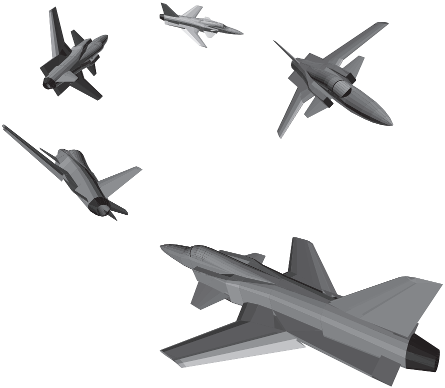


Outline

Introduction to Affine Maps in 3D

Affine maps in 3D: fighter jets twisting and turning through 3D space

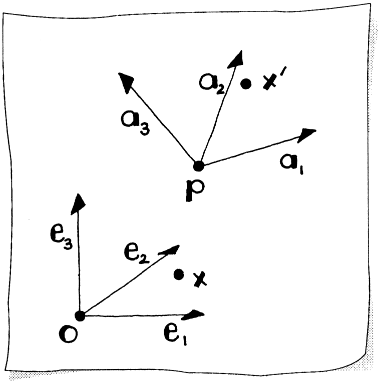


Affine maps in 3D are a primary tool for modeling and computer graphics

Additional topic in this chapter: projective maps
— the maps used to create realistic 3D images
— not affine maps but an important class of maps

Affine Maps

An affine map in 3D



Linear maps relate vectors to vectors
Affine maps relate points to points

A 3D affine map:

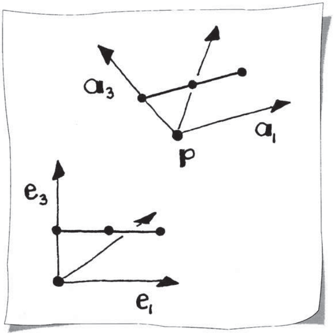
$$\mathbf{x}' = \mathbf{p} + A(\mathbf{x} - \mathbf{o})$$

where $\mathbf{x}, \mathbf{o}, \mathbf{p}, \mathbf{x}'$ are 3D points and
A is a 3×3 matrix
In general: assume that the origin of
$\mathbf{o} = \mathbf{0}$ and drop it — resulting in

$$\mathbf{x}' = \mathbf{p} + A\mathbf{x}$$

Affine Maps

Property of affine maps:

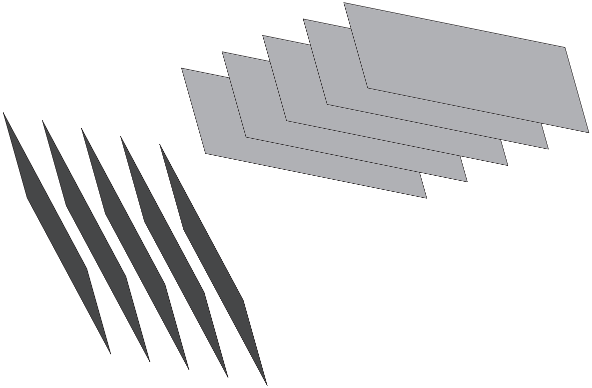


Affine maps leave ratios invariant

This map is a rigid body motion

Affine Maps

Property of affine maps:

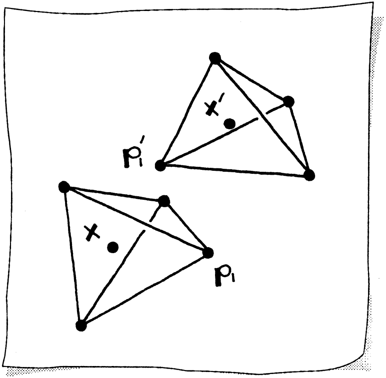


Affine maps take *parallel planes* to parallel planes

Affine maps take *intersecting planes* to intersecting planes
— the intersection line of the mapped planes is the map of the original intersection line

Affine Maps

Property of affine maps:



Affine maps leave *barycentric combinations* invariant

If

$$\mathbf{x} = c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 + c_4\mathbf{p}_4$$

where $c_1 + c_2 + c_3 + c_4 = 1$
then after an affine map

$$\mathbf{x}' = c_1\mathbf{p}'_1 + c_2\mathbf{p}'_2 + c_3\mathbf{p}'_3 + c_4\mathbf{p}'_4$$

Example: the *centroid* of a tetrahedron ⇒ centroid of the mapped tetrahedron

A translation is simply an affine map

$$\mathbf{x}' = \mathbf{p} + A(\mathbf{x} - \mathbf{o})$$

with $A = I$ (3×3 *identity matrix*)
Commonly: $\mathbf{o} = \mathbf{0}$
$[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$-system has coordinate axes parallel to $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system

Translation is a rigid body motion
— Volume of object not changed

Mapping Tetrahedra

A 3D affine map is determined by four point pairs

$$\mathbf{p}_i \rightarrow \mathbf{p}'_i \quad \text{for } i = 1, 2, 3, 4$$

$\Rightarrow$ map determined by a tetrahedron and its image
What is the image of an arbitrary point $\mathbf{x}$ under this affine map?

Key: affine maps leave *barycentric combinations* unchanged

$$\mathbf{x} = u_1\mathbf{p}_1 + u_2\mathbf{p}_2 + u_3\mathbf{p}_3 + u_4\mathbf{p}_4 \qquad (*)$$
$$\mathbf{x}' = u_1\mathbf{p}'_1 + u_2\mathbf{p}'_2 + u_3\mathbf{p}'_3 + u_4\mathbf{p}'_4$$

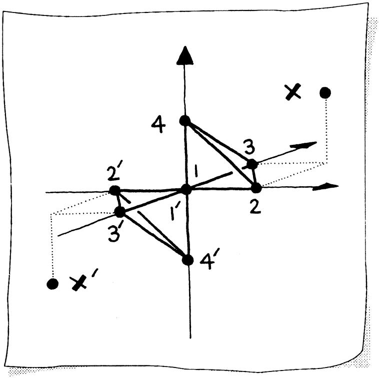
Must find the u_i — the *barycentric coordinates* of $\mathbf{x}$ with respect to the $\mathbf{p}_i$

Linear system: three coordinate equations from (*) and one from
$u_1 + u_2 + u_3 + u_4 = 1 \Rightarrow$ four equations for the four unknowns $u_1, \ldots, u_4$

(This problem is analogous to 2D case with a triangle)

Mapping Tetrahedra

Example:



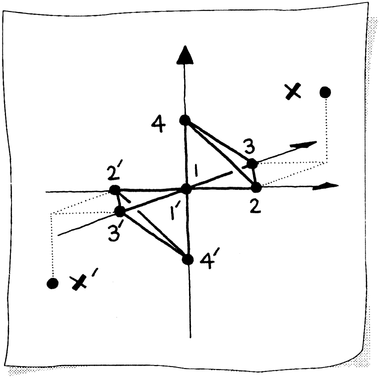
Original tetrahedron be given by $\mathbf{p}_i$

$$\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Map this tetrahedron to points $\mathbf{p}'_i$

$$\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}-1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\-1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\-1\end{bmatrix}$$

Where is point $\mathbf{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ mapped?



For this simple example we see that

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = -2\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Barycentric coordinates of $\mathbf{x}$ with respect to the original $\mathbf{p}_i$ are $(-2, 1, 1, 1)$
— Note: they sum to one

$$\mathbf{x}' = -2\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}-1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix}$$

$$= \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$$

Mapping Tetrahedra

A different approach to constructing the affine map:

Construct a coordinate system from the $\mathbf{p}_i$
— Choose $\mathbf{p}_1$ as the origin
— Three axes are defined as $\mathbf{p}_i - \mathbf{p}_1 \;\; i = 2, 3, 4$
Coordinate system of the $\mathbf{p}'_i$ based on the same indices
Find the 3×3 matrix A and point $\mathbf{p}$ that describe the affine map

$$\mathbf{x}' = A[\mathbf{x} - \mathbf{p}_1] + \mathbf{p}'_1 \qquad \rightarrow \mathbf{p} = \mathbf{p}'_1$$

We know that:

$$A[\mathbf{p}_2 - \mathbf{p}_1] = \mathbf{p}'_2 - \mathbf{p}'_1 \qquad A[\mathbf{p}_3 - \mathbf{p}_1] = \mathbf{p}'_3 - \mathbf{p}'_1 \qquad A[\mathbf{p}_4 - \mathbf{p}_1] = \mathbf{p}'_4 - \mathbf{p}'_1$$

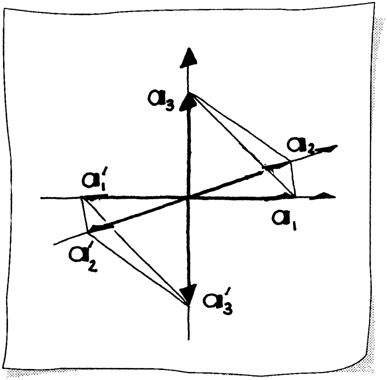
Written in matrix form:

$$A\begin{bmatrix}\mathbf{p}_2 - \mathbf{p}_1 & \mathbf{p}_3 - \mathbf{p}_1 & \mathbf{p}_4 - \mathbf{p}_1\end{bmatrix} = \begin{bmatrix}\mathbf{p}'_2 - \mathbf{p}'_1 & \mathbf{p}'_3 - \mathbf{p}'_1 & \mathbf{p}'_4 - \mathbf{p}'_1\end{bmatrix}$$

$$A = \begin{bmatrix}\mathbf{p}'_2 - \mathbf{p}'_1 & \mathbf{p}'_3 - \mathbf{p}'_1 & \mathbf{p}'_4 - \mathbf{p}'_1\end{bmatrix}\begin{bmatrix}\mathbf{p}_2 - \mathbf{p}_1 & \mathbf{p}_3 - \mathbf{p}_1 & \mathbf{p}_4 - \mathbf{p}_1\end{bmatrix}^{-1}$$

Mapping Tetrahedra

Revisiting the previous example



Select $\mathbf{p}_1$ as the origin for the $\mathbf{p}_i$ coordinate system

— since $\mathbf{p}_1 = \mathbf{0} \Rightarrow$ no translation

Compute A:

$$\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{array}$$

$$\mathbf{x}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

Same as barycentric coordinate approach

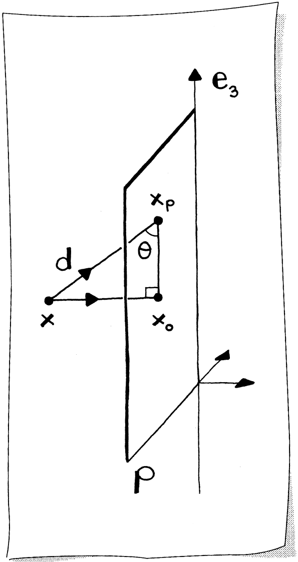
Parallel Projections

Projections in 3D: a 3D helix is projected into two different 2D planes

Earlier: looked at orthogonal parallel projections as basic linear maps

Everything we draw is a projection of necessity
—paper is 2D after all

Next: projections in the context of 3D affine maps
— maps 3D points onto a plane

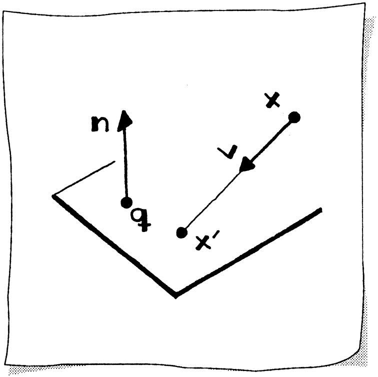


x is projected to $\mathbf{x}_p$
Parallel projection: defined by
— *direction* of projection **d** and
— *projection plane P*

Defines a *projection angle* θ
between **d** and line joining $\mathbf{x}_o$ in P
Angle categorizes parallel projections
—*orthogonal* or *oblique*

Orthogonal (orthographic)
projections are special
— **d** perpendicular to the plane
Special names for many projection
angles

Projecting a point on a plane



Project $\mathbf{x}$ in direction $\mathbf{v}$

Projection plane $[\mathbf{x}' - \mathbf{q}] \cdot \mathbf{n} = 0$

Find the projected point $\mathbf{x}' = \mathbf{p} + t\mathbf{v}$

$\Rightarrow$ find t

$$[\mathbf{x} + t\mathbf{v} - \mathbf{q}] \cdot \mathbf{n} = 0$$

$$[\mathbf{x} - \mathbf{q}] \cdot \mathbf{n} + t\mathbf{v} \cdot \mathbf{n} = 0$$

$$t = \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}$$

Intersection point $\mathbf{x}'$ computed as

$$\mathbf{x}' = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v}$$

Parallel Projections

How to write

$$\mathbf{x}' = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v}$$

as an affine map in the form $A\mathbf{x} + \mathbf{p}$?

$$\mathbf{x}' = \mathbf{x} - \frac{\mathbf{n} \cdot \mathbf{x}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v}$$

Write dot products in matrix form:

$$\mathbf{x}' = \mathbf{x} - \frac{\mathbf{n}^{\mathrm{T}} \mathbf{x}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v}$$

Observe that $\left[\mathbf{n}^{\mathrm{T}} \mathbf{x}\right] \mathbf{v} = \mathbf{v} \left[\mathbf{n}^{\mathrm{T}} \mathbf{x}\right]$

Matrix multiplication is associative: $\mathbf{v} \left[\mathbf{n}^{\mathrm{T}} \mathbf{x}\right] = \left[\mathbf{v}\mathbf{n}^{\mathrm{T}}\right] \mathbf{x}$

— Notice that $\mathbf{v}\mathbf{n}^{\mathrm{T}}$ is a 3×3 matrix

$$\mathbf{x}' = \left[I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v} \cdot \mathbf{n}} \right] \mathbf{x} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v}$$

Achieved form $\mathbf{x}' = A\mathbf{x} + \mathbf{p}$

Parallel Projections

Check the properties of

$$\mathbf{x}' = \left[I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\cdot\mathbf{n}}\right]\mathbf{x} + \frac{\mathbf{q}\cdot\mathbf{n}}{\mathbf{v}\cdot\mathbf{n}}\mathbf{v} \qquad \text{where } A = I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\cdot\mathbf{n}}$$

Projection matrix A has rank two $\Rightarrow$ reduces dimensionality

Map is idempotent:

$$\begin{aligned} A^2 &= (I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}})(I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}}) \\ &= I^2 - 2\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}} + (\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}})^2 \\ &= A - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}} + (\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}})^2 \end{aligned}$$

Expanding the squared term

$$\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}\mathbf{v}\mathbf{n}^{\mathrm{T}}}{(\mathbf{v}\cdot\mathbf{n})^2} = \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\cdot\mathbf{n}}$$

and thus $A^2 = A$

Parallel Projections

Repeating the affine map is idempotent as well:

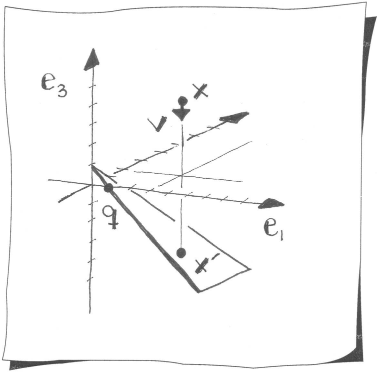
$$A(A\mathbf{x} + \mathbf{p}) + \mathbf{p} = A^2\mathbf{x} + A\mathbf{p} + \mathbf{p}$$
$$= A\mathbf{x} + A\mathbf{p} + \mathbf{p}$$

Let $\alpha = (\mathbf{q} \cdot \mathbf{n})/(\mathbf{v} \cdot \mathbf{n})$ — examine the middle term

$$A\mathbf{p} = (I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}})\alpha\mathbf{v}$$
$$= \alpha\mathbf{v} - \alpha\mathbf{v}(\frac{\mathbf{n}^{\mathrm{T}}\mathbf{v}}{\mathbf{v}\mathbf{n}})$$
$$= 0$$

$\Rightarrow A(A\mathbf{x} + \mathbf{p}) + \mathbf{p} = A\mathbf{x} + \mathbf{p}$
$\Rightarrow$ the affine map is idempotent

Example:



Given:

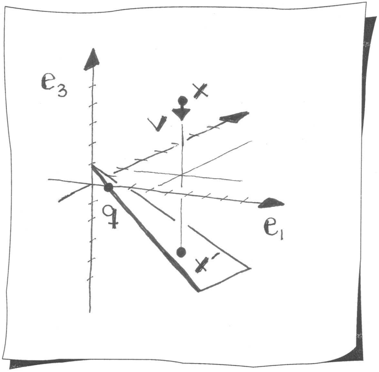
projection plane $x_1 + x_2 + x_3 - 1 = 0$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \quad \text{direction } \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Project **x** along **v** onto the plane: what is $\mathbf{x}'$?

Plane's normal $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Parallel Projections



Choose point $\mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the plane

Calculate needed quantities:

$$\mathbf{v} \cdot \mathbf{n} = -1 \qquad \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\mathbf{v}\mathbf{n}^{\mathrm{T}} =$

	1	1	1
0	0	0	0
0	0	0	0
−1	−1	−1	−1

Putting all the pieces together:

$$\mathbf{x}' = \left[I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right] \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

Homogeneous Coordinates and Perspective Maps

Homogeneous matrix form:

Condense $\mathbf{x}' = A\mathbf{x} + \mathbf{p}$ into just one matrix multiplication $\underline{\mathbf{x}}' = M\underline{\mathbf{x}}$

$$\underline{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \qquad \underline{\mathbf{x}}' = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ 1 \end{bmatrix} \qquad M = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & p_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & p_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4D point $\underline{\mathbf{x}}$ is the homogeneous form of the affine point $\mathbf{x}$

Homogeneous representation of a vector $\underline{\mathbf{v}} = [v_1 \quad v_2 \quad v_3 \quad 0]^{\mathrm{T}}$

$\Rightarrow \underline{\mathbf{v}}' = M\underline{\mathbf{v}}$

Zero fourth component $\Rightarrow$ disregard the translation

— translation has no effect on vectors

— vector defined as the difference of two points

Homogeneous Coordinates and Perspective Maps

Advantage of the homogeneous matrix form:
Condenses information into one matrix
— Implemented in the popular computer graphics *Application Programmer's Interface*
— Convenient and efficient to have all information in one data structure

Homogeneous point $\underline{\mathbf{x}}$ to affine counterpart $\mathbf{x}$: divide through by x_4
$\Rightarrow$ one affine point has infinitely many homogeneous representations

Example:

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \approx \begin{bmatrix} 10 \\ -10 \\ 30 \\ 10 \end{bmatrix} \approx \begin{bmatrix} -2 \\ 2 \\ -6 \\ -2 \end{bmatrix}$$

(Symbol $\approx$ should be read "corresponds to.")

Revisit a projection problem:
Given point $\mathbf{x}$, projection direction $\mathbf{v}$, and projection plane $[\mathbf{x}' - \mathbf{q}] \cdot \mathbf{n} = 0$
The projected point

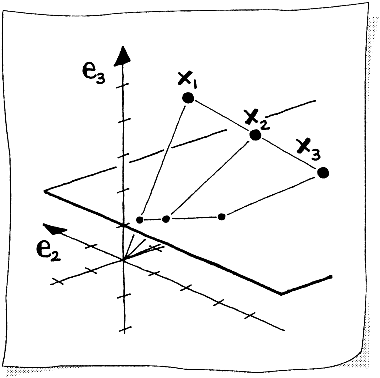
$$\mathbf{x}' = \left[I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v} \cdot \mathbf{n}} \right] \mathbf{x} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v}$$

The homogeneous matrix form:

$$\left[\begin{array}{c|c} \begin{bmatrix} \mathbf{v}\cdot\mathbf{n} & 0 & 0 \\ 0 & \mathbf{v}\cdot\mathbf{n} & 0 \\ 0 & 0 & \mathbf{v}\cdot\mathbf{n} \end{bmatrix} - \mathbf{v}\mathbf{n}^{\mathrm{T}} & (\mathbf{q}\cdot\mathbf{n})\mathbf{v} \\ \hline 0 \quad 0 \quad 0 & \mathbf{v}\cdot\mathbf{n} \end{array}\right]$$

Homogeneous Coordinates and Perspective Maps

Perspective projection



Instead of a constant direction $\mathbf{v}$
Perspective projection direction depends on the point $\mathbf{x}$ — the line from $\mathbf{x}$ to the origin: $\mathbf{v} = -\mathbf{x}$

$$\mathbf{x}' = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{x} \cdot \mathbf{n}} \mathbf{x} = \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{x} \cdot \mathbf{n}} \mathbf{x}$$

Homogeneous matrix form:

$$M : \left[\begin{array}{ccc|c} & I[\mathbf{q} \cdot \mathbf{n}] & & \mathbf{o} \\ \hline 0 & 0 & 0 & \mathbf{x} \cdot \mathbf{n} \end{array}\right]$$

Homogeneous Coordinates and Perspective Maps

Perspective projections are not affine maps

Example: Plane $x_3 = 1$ and point on the plane $\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\mathbf{q} \cdot \mathbf{n} = 1$ and $\mathbf{x} \cdot \mathbf{n} = x_3$ — resulting in the map $\mathbf{x}' = \frac{1}{x_3}\mathbf{x}$

Take the three points (see previous Sketch)

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

Note: $\mathbf{x}_2$ is the midpoint of $\mathbf{x}_1$ and $\mathbf{x}_3$ Their images are

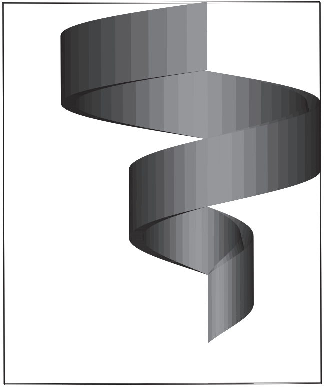
$$\mathbf{x}'_1 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{x}'_2 = \begin{bmatrix} 1 \\ -1/3 \\ 1 \end{bmatrix} \quad \mathbf{x}'_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The perspective map destroyed the midpoint relation

$$\mathbf{x}'_2 = \frac{2}{3}\mathbf{x}'_1 + \frac{1}{3}\mathbf{x}'_3$$

Homogeneous Coordinates and Perspective Maps

Perspective maps
— do not preserve the ratio of three points
— two parallel lines will not be mapped to parallel lines
— good model for how we perceive 3D space around us



Left: Parallel projection   Right: Perspective projection

Homogeneous Coordinates and Perspective Maps

Experiment by A. Dürer
From *The Complete Woodcuts of Albrecht Dürer,* edited by W. Durth, Dover Publications Inc., NY, 1963

Study of perspective goes back to the 14th century
Earlier times:
artists could not draw realistic 3D images

WYSK

- affine map
- translation
- affine map properties
- barycentric combination
- invariant ratios
- barycentric coordinates
- centroid
- mapping four points to four points
- parallel projection
- orthogonal projection
- oblique projection
- line and plane intersection
- idempotent
- dyadic matrix
- homogeneous coordinates
- perspective projection
- rank