

# Practical Linear Algebra: A GEOMETRY TOOLBOX

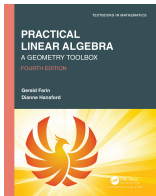
Fourth Edition

## Chapter 3: Lining Up: 2D Lines

Gerald Farin & Dianne Hansford

A K Peters/CRC Press  
[www.farinhanford.com/books/pla](http://www.farinhanford.com/books/pla)

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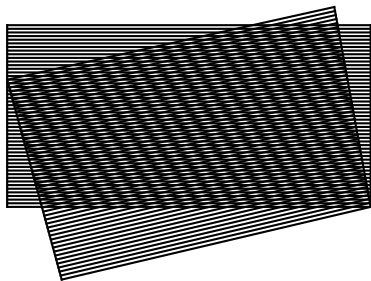


# Outline

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# Introduction to 2D Lines

2D lines are the building blocks for many geometric constructions



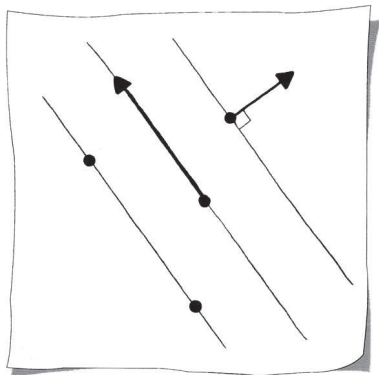
Two sets of parallel lines overlaid  
Interference pattern called  
*Moiré pattern*  
Used in optics for checking the  
properties of lenses

Chapter focus:

- representations for lines
- distance from a line
- intersections

# Defining a Line

Two elements of 2D geometry define a line:



Elements can be

- two points
- a point and a vector parallel to the line
- a point and a vector perpendicular to the line

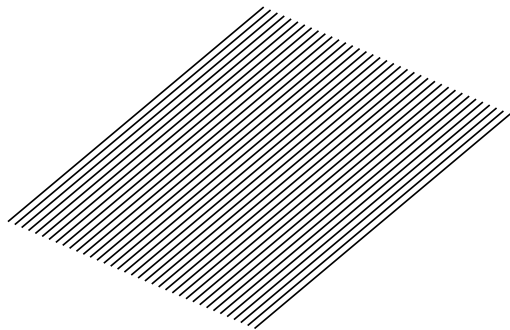
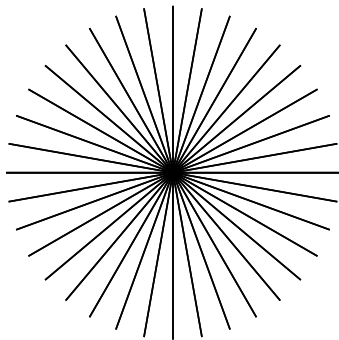
**Normal** to a line: unit vector that is perpendicular (or orthogonal) to a line

# Defining a Line

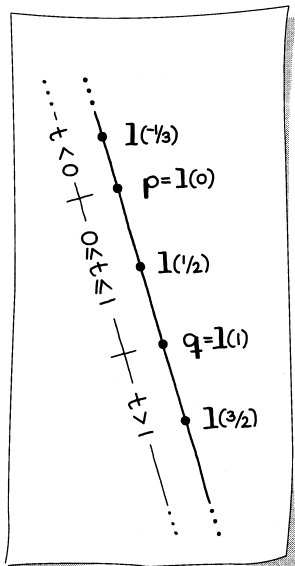
## Families of lines

One family of lines shares a common point

The other family of lines shares the same normal



# Parametric Equation of a Line



$$\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$$

where  $\mathbf{p} \in \mathbb{E}^2$  and  $\mathbf{v} \in \mathbb{R}^2$

Scalar value  $t$  is the **parameter**

Evaluating for specific parameter generates a point on the line

# Parametric Equation of a Line

Parametric form  $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$   
in terms of barycentric coordinates:

Let  $\mathbf{v} = \mathbf{q} - \mathbf{p}$ , then

$$\mathbf{l}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$$

$(1 - t)$  and  $t$  are the barycentric coordinates  
of a point on the line with respect to  $\mathbf{p}$  and  $\mathbf{q}$ , resp.

Typically referred to as **linear interpolation**

# Parametric Equation of a Line

**Barycentric combination:** sum of coefficients of points is one

$$\mathbf{l}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$$

For  $t \in [0, 1]$ : generates points on line between  $\mathbf{p}$  and  $\mathbf{q}$

This is a **convex combination**

$t < 0$  generates points on the line “behind”  $\mathbf{p}$

$t > 1$  generates points “past”  $\mathbf{q}$

(Interpret as scaling of  $\mathbf{v}$  in  $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$ )

This is **extrapolation**



# Parametric Equation of a Line

Parametric form very good for computing points on a line

## Example:

Compute ten equally spaced points on line segment through  $\mathbf{p}$  and  $\mathbf{q}$

$$\mathbf{l}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$$

$$t = i/9, \quad i = 0, \dots, 9$$

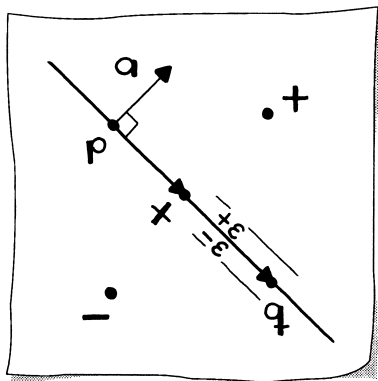
$$i = 0 \quad \rightarrow \quad t = 0 \quad \rightarrow \quad \mathbf{p}$$

$$i = 9 \quad \rightarrow \quad t = 1 \quad \rightarrow \quad \mathbf{q}$$

Equally spaced parameter values correspond to equally spaced points

**Parametrization:** speed line is traversed

# Implicit Equation of a Line



Point  $\mathbf{p}$  and a vector  $\mathbf{a}$  perpendicular to the line For any point  $\mathbf{x}$  on the line

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

$\mathbf{a}$  and  $(\mathbf{x} - \mathbf{p})$  are perpendicular

If  $\mathbf{a}$  unit length,  
called **point normal form**

Expand:

$$a_1x_1 + a_2x_2 + (-a_1p_1 - a_2p_2) = 0$$

Results in familiar form:

$$ax_1 + bx_2 + c = 0$$

# Implicit Equation of a Line

**Given:** two points on the line

$$\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

**Find:** implicit line  $ax_1 + bx_2 + c = 0$

$$\mathbf{v} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

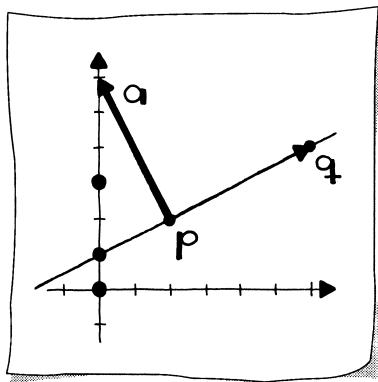
$$\mathbf{a} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$c = 2 \times 2 - 4 \times 2 = -4$$

Implicit equation of the line:

$$-2x_1 + 4x_2 - 4 = 0$$

(Not point normal form)



# Implicit Equation of a Line

Implicit form good for deciding if an arbitrary point lies on the line

For given point  $x$

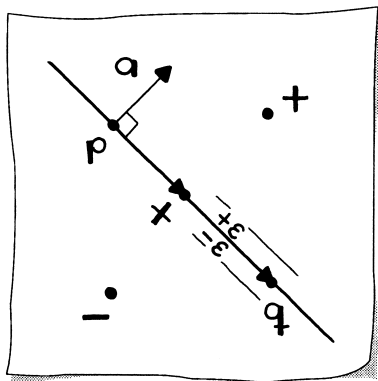
$$f = ax_1 + bx_2 + c$$

If  $f = 0$  then the point is on the line

# Implicit Equation of a Line

Numerical caveat:

Checking equality  $f = 0.0$  with floating point numbers not recommended



Tolerance  $\epsilon$  needed

Meaningful tolerance:

true distance of  $x$  to line

$$d = \frac{f}{\|\mathbf{a}\|}$$

Sign of  $d$  indicates side of the line

# Explicit Equation of a Line

Explicit form is closely related to the implicit form  $ax_1 + bx_2 + c = 0$

Expresses  $x_2$  as a function of  $x_1$

$$x_2 = -\frac{a}{b}x_1 - \frac{c}{b}$$

Simpler:  $x_2 = \hat{a}x_1 + \hat{b}$

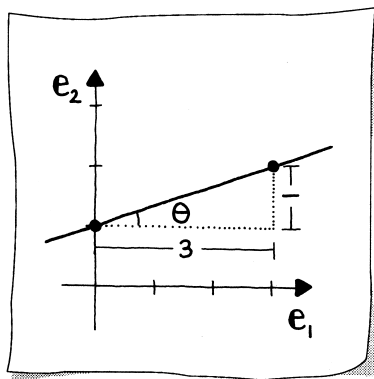
Geometric meaning of coefficients:

$\hat{a}$  is the **slope** “rise/run”

$\hat{b}$  is the  $e_2$ -intercept

**Example:**  $x_2 = \frac{1}{3}x_1 + 1$

Drawback: vertical line has infinite slope



# Converting Between Parametric and Implicit Equations

Advantages to both the parametric and implicit representations of a line

Depending on the geometric algorithm  
may be convenient to use one form rather than the other

Ignore the explicit form:  
not very useful for general 2D geometry

# Converting Between Parametric and Implicit Equations

Parametric to Implicit

**Given:**  $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$

**Find:** coefficients  $a, b, c$  of  $ax_1 + bx_2 + c = 0$

**Solution:** Form a vector  $\mathbf{a} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$  perpendicular to vector  $\mathbf{v}$

Determines the coefficients  $a = a_1$  and  $b = a_2$

Use  $\mathbf{p}$  from  $\mathbf{l}(t)$  to compute  $c = -(a_1p_1 + a_2p_2)$



# Converting Between Parametric and Implicit Equations

Implicit to Parametric

**Given:**  $ax_1 + bx_2 + c = 0$

**Find:**  $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$

**Solution:** Need one point on the line and a vector parallel to the line

Vector  $\mathbf{v} = \begin{bmatrix} b \\ -a \end{bmatrix}$  is perpendicular to  $\mathbf{a}$

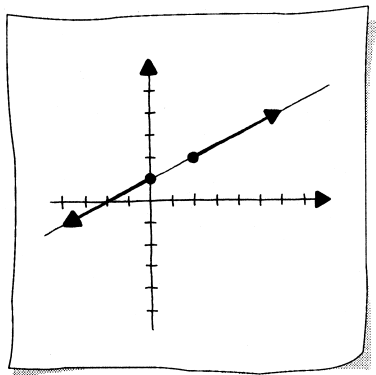
Point candidates: intersections with the  $\mathbf{e}_1$ - or  $\mathbf{e}_2$ -axis

$$\begin{bmatrix} -c/a \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ -c/b \end{bmatrix}$$

For numerical stability, choose the intersection closest to the origin

# Converting Between Parametric and Implicit Equations

## Non-uniqueness of representations



Two parametric representations for the same line  
Lines will be *traced* differently

Conversion process – for example  
parametric  $\rightarrow$  implicit  $\rightarrow$  parametric  
Original and final parametric form  
different in general

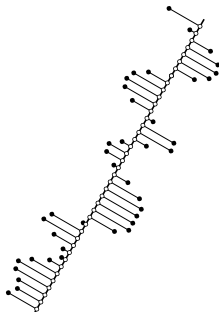
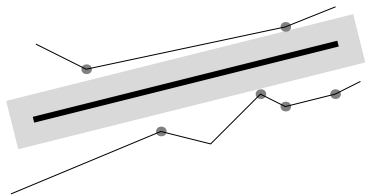
# Distance of a Point to a Line

Robot will travel along the line (thick black)

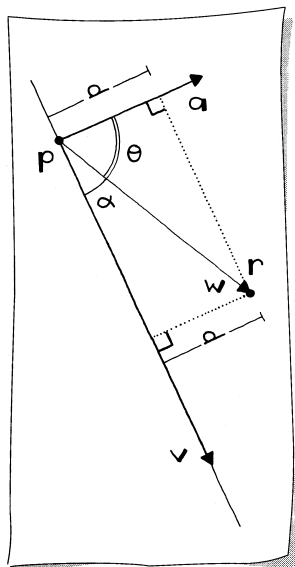
Points represent objects in the room

Verify needed robot clearance (gray)  
by computing distance of point to line

Smallest distance  $d(\mathbf{r}, \mathbf{l})$   
of a point to a line is the  
orthogonal or perpendicular distance



# Distance of a Point to a Line



Starting with an Implicit Line

**Given:**  $l : ax_1 + bx_2 + c$  and point  $r$

**Find:**  $d(r, l)$

**Solution:** let  $\mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$d = \frac{ar_1 + br_2 + c}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot (\mathbf{r} - \mathbf{p})}{\|\mathbf{a}\|}$$

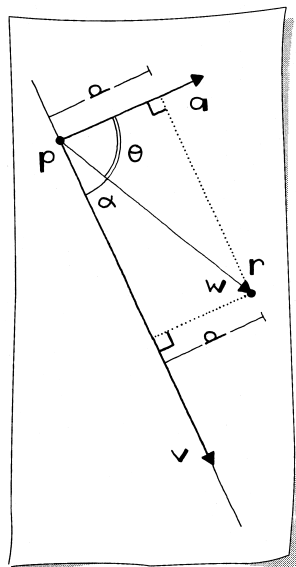
Let  $\mathbf{w} = \mathbf{r} - \mathbf{p}$

$$v = \mathbf{a} \cdot \mathbf{w} = \|\mathbf{a}\| \|\mathbf{w}\| \cos(\theta)$$

$$\cos(\theta) = \frac{d}{\|\mathbf{w}\|} \quad \text{then} \quad v = \|\mathbf{a}\| d$$

$$d = \frac{ar_1 + br_2 + c}{\|\mathbf{a}\|}$$

# Distance of a Point to a Line



Starting with a Parametric Line

**Given:**  $l(t) = \mathbf{p} + t\mathbf{v}$  and a point  $\mathbf{r}$

**Find:**  $d(\mathbf{r}, l)$

**Solution:** Let  $\mathbf{w} = \mathbf{r} - \mathbf{p}$

$$d = \|\mathbf{w}\| \sin(\alpha)$$

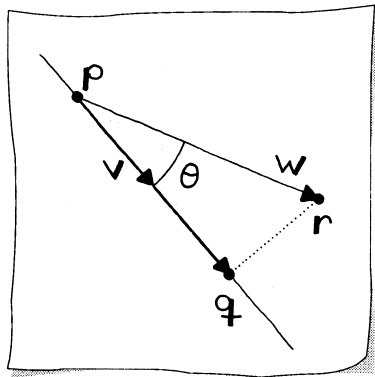
$$\sin(\alpha) = \sqrt{1 - \cos(\alpha)^2}$$

$$\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

# The Foot of a Point

Which point on the line is closest to a given point?

This point is the **foot** of the given point



**Given:**  $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$  and point  $\mathbf{r}$

**Find:**  $\mathbf{q}$ , the foot of  $\mathbf{r}$

**Solution:**  $t$  such that  $\mathbf{q} = \mathbf{p} + t\mathbf{v}$

Define  $\mathbf{w} = \mathbf{r} - \mathbf{p}$

$$\cos(\theta) = \frac{\|t\mathbf{v}\|}{\|\mathbf{w}\|} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

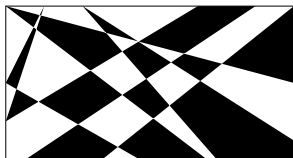
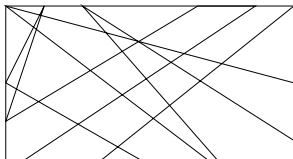
Solve for  $t$

$$t = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}$$

# A Meeting Place: Computing Intersections

Intersection problems important in many applications

Fun figure: finding intersections to create an artistic image



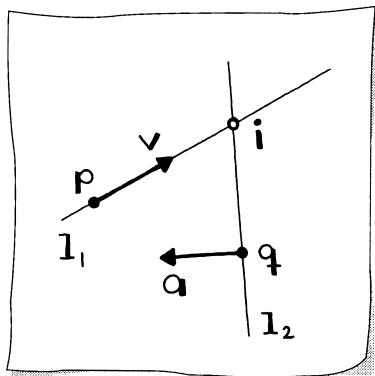
Possible questions:

- Do the lines intersect?
- At what point do they intersect?
- At what parameter value on one or both lines is the intersection point?

Question(s) + line representation(s)  
⇒ best method

# A Meeting Place: Computing Intersections

## Parametric and Implicit



**Given:**

$$l_1(t) = \mathbf{p} + t\mathbf{v}$$

$$l_2 : ax_1 + bx_2 + c = 0$$

**Find:** intersection point  $\mathbf{i}$

**Solution:**  $\mathbf{i} = \mathbf{p} + \hat{t}\mathbf{v}$

$$a[p_1 + \hat{t}v_1] + b[p_2 + \hat{t}v_2] + c = 0$$

One equation and one unknown

$$\hat{t} = \frac{-c - ap_1 - bp_2}{av_1 + bv_2}$$

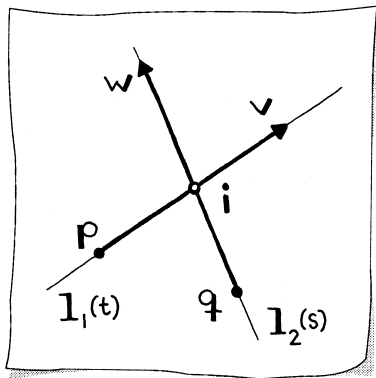
$$\mathbf{i} = l_1(\hat{t})$$

What if the lines are parallel?



# A Meeting Place: Computing Intersections

Both Parametric



**Given:**

$$l_1(t) = \mathbf{p} + t\mathbf{v} \quad l_2(s) = \mathbf{q} + s\mathbf{w}$$

**Find:** intersection point  $\mathbf{i}$

**Solution:**  $\hat{t}$  and  $\hat{s}$  such that

$$\mathbf{p} + \hat{t}\mathbf{v} = \mathbf{q} + \hat{s}\mathbf{w}.$$

Rewritten:

$$\hat{t}\mathbf{v} - \hat{s}\mathbf{w} = \mathbf{q} - \mathbf{p}$$

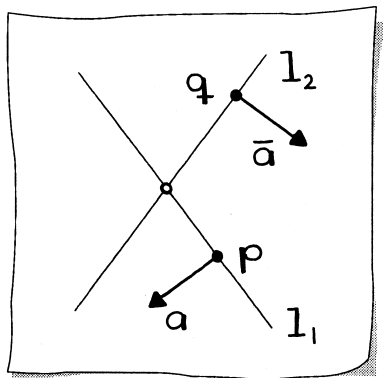
Two equations and two unknowns

If  $\mathbf{v}$  and  $\mathbf{w}$  linearly dependent: no solution

Geometric meaning?

# A Meeting Place: Computing Intersections

Both Implicit



**Given:**

$$l_1 : ax_1 + bx_2 + c = 0$$

$$l_2 : \bar{a}x_1 + \bar{b}x_2 + \bar{c} = 0$$

**Find:** intersection point  $\mathbf{i} = \hat{\mathbf{x}}$   
that satisfies  $l_1$  and  $l_2$

**Solution:**

$$a\hat{x}_1 + b\hat{x}_2 = -c$$

$$\bar{a}\hat{x}_1 + \bar{b}\hat{x}_2 = -\bar{c}$$

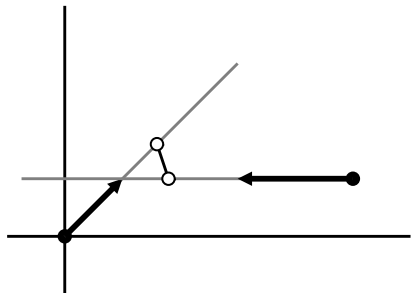
Two equations and two unknowns

Lines parallel

$\Rightarrow \mathbf{a}$  and  $\bar{\mathbf{a}}$  linearly dependent

# Application: Closest Point of Approach

Two ships travelling at constant speed along linear paths  
At what position and time will the two ships be closest?



Ship path defined by an initial position and velocity vector:

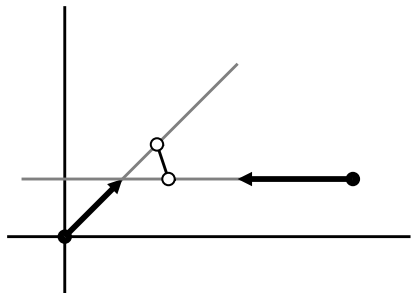
$$\mathbf{c}(t) = \mathbf{c}_0 + t\mathbf{v}_c \quad \mathbf{m}(t) = \mathbf{m}_0 + t\mathbf{v}_m$$

Example in the Figure:

$$\mathbf{c}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

$$\mathbf{m}(t) = \begin{bmatrix} 100 \\ 20 \end{bmatrix} + t \begin{bmatrix} -40 \\ 0 \end{bmatrix}$$

# Application: Closest Point of Approach



Form vector from corresponding points on each path at any time  $t$

$$\mathbf{v}(t) = \mathbf{c}(t) - \mathbf{m}(t) = \mathbf{v}_0 + t(\mathbf{v}_v)$$

$$\mathbf{v}_0 = \mathbf{c}_0 - \mathbf{m}_0 \text{ and } \mathbf{v}_v = \mathbf{v}_c - \mathbf{v}_m$$

At what time  $t$  is  $\|\mathbf{v}(t)\|$  minimized?

Easier (equivalent) problem:

At what time  $t$  is  $\|\mathbf{v}(t)\|^2$  minimized?

Let

$$d = \|\mathbf{v}(t)\|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t)$$

## Application: Closest Point of Approach

Find the minimum of the quadratic polynomial

$$d(t) = \mathbf{v}_0 \cdot \mathbf{v}_0 + 2(\mathbf{v}_0 \cdot \mathbf{v}_v)t + (\mathbf{v}_v \cdot \mathbf{v}_v)t^2$$

Take the derivative and find where the function is equal to zero

$$\frac{dd(t)}{dt} = 2(\mathbf{v}_0 \cdot \mathbf{v}_v) + 2(\mathbf{v}_v \cdot \mathbf{v}_v)t = 0$$

Solving for  $t$

$$t = \frac{\mathbf{v}_0 \cdot \mathbf{v}_v}{\|\mathbf{v}_v\|^2}$$

In our ship example,

$$\mathbf{v}_0 = \begin{bmatrix} -100 \\ -20 \end{bmatrix}, \quad \mathbf{v}_v = \begin{bmatrix} 60 \\ 20 \end{bmatrix}, \quad \text{then } t = 1.6$$

Closest point of approach:

$$\mathbf{c}(1.6) = \begin{bmatrix} 32 \\ 32 \end{bmatrix} \quad \text{and} \quad \mathbf{m}(1.6) = \begin{bmatrix} 36 \\ 20 \end{bmatrix}$$

- parametric form of a line
- linear interpolation
- point normal form
- implicit form of a line
- explicit form of a line
- line through two points
- line defined by a point and a vector parallel to the line
- line defined by a point and a vector perpendicular to the line
- distance of a point to a line
- line form conversions
- foot of a point
- intersection of lines