Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 4: Changing Shapes: Linear Maps in 2D

Gerald Farin & Dianne Hansford

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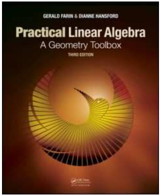

Outline

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2. Skew Target Boxes
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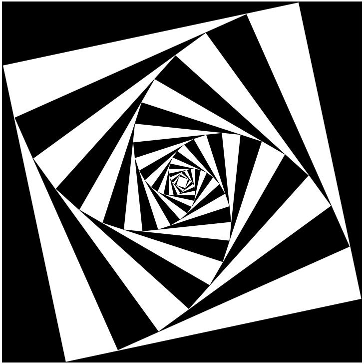
2D linear maps (rotation and scaling) applied repeatedly to a square



Geometry has two parts

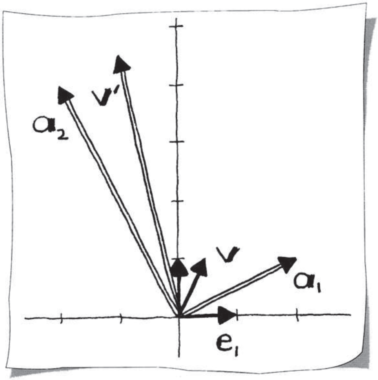
1. description of the objects
2. how these objects can be changed (transformed)

Transformations also called maps
May be described using the tools of matrix operations: linear maps

Matrices first introduced by H. Grassmann in 1844
Became basis of linear algebra

Skew Target Boxes

Revisit unit square to a rectangular target box mapping
Examine part of mapping that is a linear map



Unit square defined by $\mathbf{e}_1$ and $\mathbf{e}_2$
Vector $\mathbf{v}$ in $[\mathbf{e}_1, \mathbf{e}_2]$-system defined as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$$

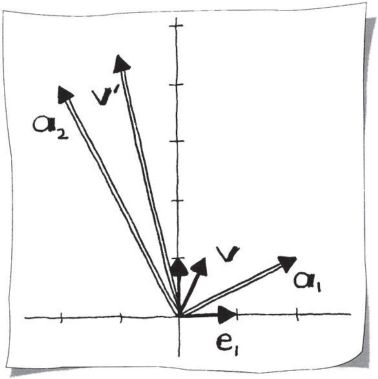
$\mathbf{v}$ is now mapped to a vector $\mathbf{v}'$ by

$$\mathbf{v}' = v_1\mathbf{a}_1 + v_2\mathbf{a}_2$$

Duplicates the $[\mathbf{e}_1, \mathbf{e}_2]$-geometry in the $[\mathbf{a}_1, \mathbf{a}_2]$-system

Skew Target Boxes

Example: *linear combination*



$[\mathbf{a}_1, \mathbf{a}_2]$-coordinate system: origin and

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Given $\mathbf{v} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ in $[\mathbf{e}_1, \mathbf{e}_2]$-system

$$\mathbf{v}' = \frac{1}{2} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 9/2 \end{bmatrix}$$

$\mathbf{v}'$ has components $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ with respect to $[\mathbf{a}_1, \mathbf{a}_2]$-system

$\mathbf{v}'$ has components $\begin{bmatrix} -1 \\ 9/2 \end{bmatrix}$ with respect to $[\mathbf{e}_1, \mathbf{e}_2]$-system

The Matrix Form

Components of a subscripted vector written with a double subscript

$$\mathbf{a}_1 = \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}$$

The vector component index precedes the vector subscript

Components for $\mathbf{v}'$ in $[\mathbf{e}_1, \mathbf{e}_2]$-system expressed as

$$\begin{bmatrix} -1 \\ 9/2 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Using matrix notation:

$$\begin{bmatrix} -1 \\ 9/2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

2×2 matrix: 2 rows and 2 columns
— Columns are vectors $\mathbf{a}_1$ and $\mathbf{a}_2$

The Matrix Form

In general:

$$\mathbf{v}' = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A\mathbf{v}$$

A is a 2×2 matrix
Elements $a_{1,1}$ and $a_{2,2}$ form the diagonal

$\mathbf{v}'$ is the image of $\mathbf{v}$
$\mathbf{v}$ is the pre-image of $\mathbf{v}'$

$\mathbf{v}'$ is in the range of the map
$\mathbf{v}$ is in the domain of the map

The Matrix Form

Product $A\mathbf{v}$ has two components:

$$A\mathbf{v} = \begin{bmatrix} v_1\mathbf{a}_1 + v_2\mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} v_1 a_{1,1} + v_2 a_{1,2} \\ v_1 a_{2,1} + v_2 a_{2,2} \end{bmatrix}$$

Each component obtained as a dot product between the corresponding row of the matrix and $\mathbf{v}$

Example:

$$\begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 14 \end{bmatrix}$$

Column space of A: all $\mathbf{v}'$ formed as linear combination of the columns of A

The Matrix Form

$[\mathbf{e}_1, \mathbf{e}_2]$-system can be interpreted as a matrix with columns $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$[\mathbf{e}_1, \mathbf{e}_2] \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Called the 2×2 identity matrix

Neat way to write matrix-times-vector:

$$\begin{array}{cc|c} & & 2 \\ & & 1/2 \\ \hline 2 & -2 & 3 \\ 1 & 4 & 4 \end{array}$$

Interior dimensions (both 2) must be identical
Outer dimensions (2 and 1) indicate the resulting vector or matrix size

The Matrix Form

Matrix addition:

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} \end{bmatrix}$$

Matrices must be of the same dimensions

Distributive law

$$A\mathbf{v} + B\mathbf{v} = (A + B)\mathbf{v}$$

The Matrix Form

Transpose matrix denoted by A^{T}

Formed by interchanging the rows and columns of A:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \quad \text{then} \quad A^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$

May think of a vector **v** as a matrix:

$$\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \text{then} \quad \mathbf{v}^{\mathrm{T}} = \begin{bmatrix} -1 & 4 \end{bmatrix}$$

Identities:

$$[A + B]^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$$

$$A^{\mathrm{T}^{\mathrm{T}}} = A \quad \text{and} \quad [cA]^{\mathrm{T}} = cA^{\mathrm{T}}$$

Symmetric matrix: $A = A^{\mathrm{T}}$

Example:

$$\begin{bmatrix} 5 & 8 \\ 8 & 1 \end{bmatrix}$$

No restrictions on diagonal elements
All other elements equal to element about the diagonal with reversed indices
For a 2×2 matrix: $a_{2,1} = a_{1,2}$

2×2 zero matrix:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The Matrix Form

Matrix rank: number of linearly independent column (row) vectors

For 2×2 matrix columns define an $[\mathbf{a}_1, \mathbf{a}_2]$-system
Full rank=2: $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent

Rank deficient: matrix that does not have full rank
If $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent then matrix has rank 1
Also called a *singular* matrix

Only matrix with rank zero is zero matrix

Rank of A and A^{T} are equal.

Linear Spaces

2D linear maps act on vectors in 2D linear spaces
Also known as 2D vector spaces

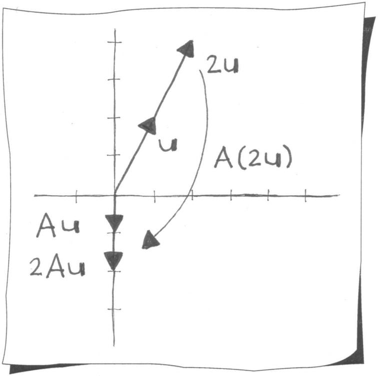
Standard operations in a linear space are addition and scalar multiplication of vectors
$\mathbf{v}' = v_1\mathbf{a}_1 + v_2\mathbf{a}_2$ — linearity property

Linear maps – matrices – characterized by preservation of linear combinations:

$$A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}.$$

Let's break this statement down into the two basic elements:
scalar multiplication and addition

Matrices preserve scalings



Example:

$$A = \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

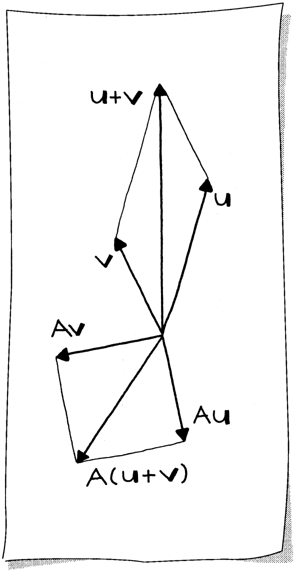
$$A(c\mathbf{u}) = cA\mathbf{u}$$

Let $c = 2$

$$\begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \left(2 \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

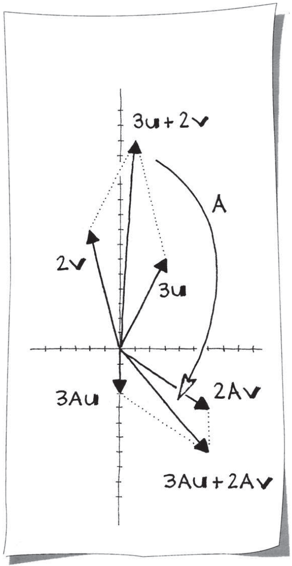
$$= 2 \times \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Linear Spaces



Matrices preserve sums (distributive law):

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$$



Matrices preserve linear combinations
$A(3\mathbf{u} + 2\mathbf{v})$

$$= \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \left(3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 6 \\ -7 \end{bmatrix} \quad 3A\mathbf{u} + 2A\mathbf{v}$$

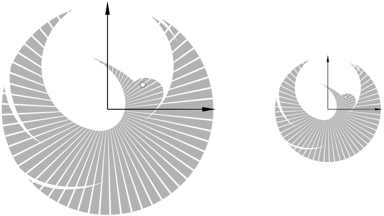
$$= 3 \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$+2 \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

Scalings

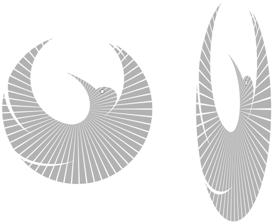
Uniform scaling:



$$\mathbf{v}' = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \mathbf{v} = \begin{bmatrix} v_1/2 \\ v_2/2 \end{bmatrix}$$

Scalings

General scaling: $\mathbf{v}' = \begin{bmatrix} s_{1,1} & 0 \\ 0 & s_{2,2} \end{bmatrix} \mathbf{v}$ **Example:** $\mathbf{v}' = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{v}$

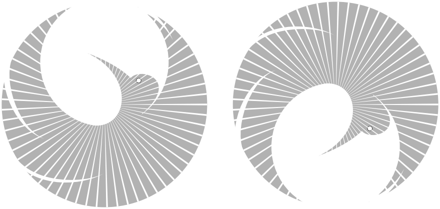


Scaling affects the *area* of the object:
— Scale by $s_{1,1}$ in $\mathbf{e}_1$-direction, then area changes by a factor $s_{1,1}$
— Similarly for $s_{2,2}$ and $\mathbf{e}_2$-direction
Total effect: factor of $s_{1,1}s_{2,2}$

Action of matrix: action ellipse

Reflections

Special scaling:



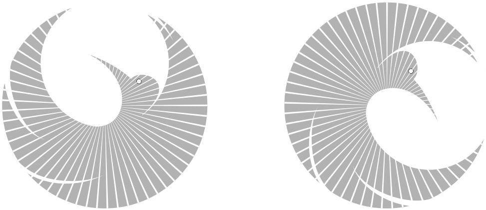
$$\mathbf{v}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

$\mathbf{v}$ reflected about $\mathbf{e}_1$-axis or the line $x_1 = 0$

Reflection maps each vector about a line through the origin

Reflections

Reflection about line $x_1 = x_2$:



$$\mathbf{v}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$$

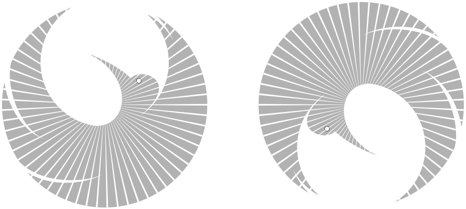
Reflections change the *sign* of the area due to a change in orientation

— Rotate $\mathbf{e}_1$ into $\mathbf{e}_2$: move in a counterclockwise

— Rotate $\mathbf{a}_1$ into $\mathbf{a}_2$: move in a clockwise

Reflections

Reflection?

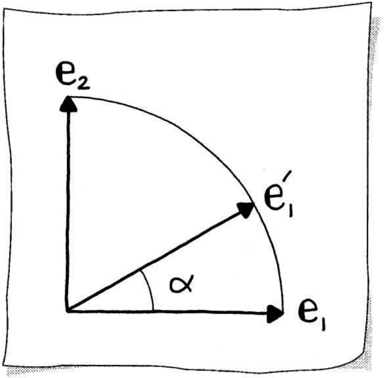


$$\mathbf{v}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}$$

Check $\mathbf{a}_1$ rotate to $\mathbf{a}_2$ orientation: counterclockwise
— same as $\mathbf{e}_1, \mathbf{e}_2$ orientation
This is a 180° rotation

Action ellipse: circle

Rotations



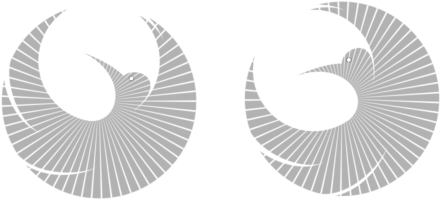
Rotate $\mathbf{e}_1$ and $\mathbf{e}_2$ around the origin to

$$\mathbf{e}_1' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$$

These are the column vectors of the rotation matrix

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Rotations



Rotation matrix for $\alpha = 45°$:

$$R = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

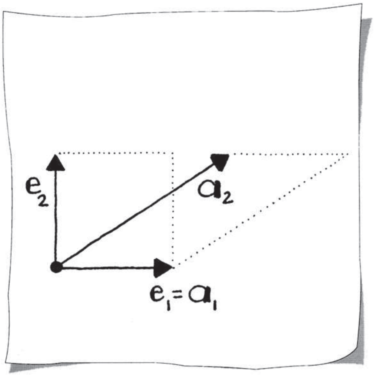
Rotations: special class of transformations called rigid body motions

Action ellipse: circle

Rotations do not change areas

Shears

Map a rectangle to a parallelogram



Example:

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \longrightarrow \quad \mathbf{v}' = \begin{bmatrix} d_1 \\ 1 \end{bmatrix}$$

A shear in matrix form:

$$\begin{bmatrix} d_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Application: generate italic fonts from standard ones.

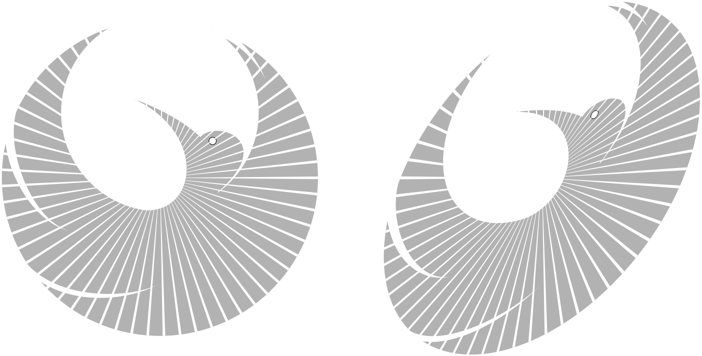
Shears

Shear along the $\mathbf{e}_1$-axis applied to an arbitrary vector:

$$\mathbf{v}' = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 d_1 \\ v_2 \end{bmatrix}$$

Shears

Shear along the $\mathbf{e}_2$-axis:



$$\mathbf{v}' = \begin{bmatrix} 1 & 0 \\ d_2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 d_2 + v_2 \end{bmatrix}$$

Shears

What is the shear that achieves

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \longrightarrow \quad \mathbf{v}' = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}?$$

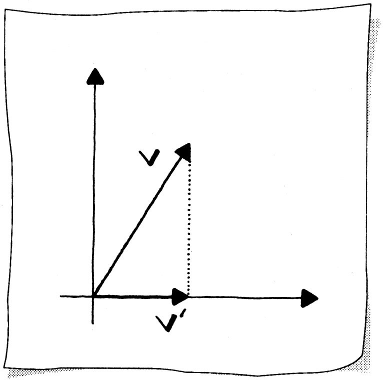
A shear parallel to the $\mathbf{e}_2$-axis:

$$\mathbf{v}' = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v_2/v_1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Shears do not change areas
(See rectangle to parallelogram sketch: both have the same base and the same height)

Projections

Parallel projections: all vectors are projected in a parallel direction
2D: all vectors are projected onto a line



Example:

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Orthogonal projection: angle of incidence with the line is 90°
Otherwise: oblique projection

Perspective projection: projection direction is not constant — not a linear map

Projections

Orthogonal projections important for best approximation
Oblique projections important to applications in fields such as computer graphics and architecture

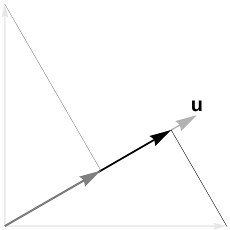
Main property of a projection: *reduces dimensionality*
Action ellipse: straight line segment which is covered twice

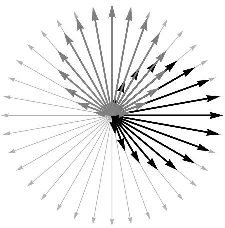
Construction:

Choose unit vector $\mathbf{u}$ to define a line onto which to project
Projections of $\mathbf{e}_1$ and $\mathbf{e}_2$ are column vectors of matrix A

$$\mathbf{a}_1 = \frac{\mathbf{u} \cdot \mathbf{e}_1}{\|\mathbf{u}\|^2}\mathbf{u} = u_1\mathbf{u}$$
$$\mathbf{a}_2 = \frac{\mathbf{u} \cdot \mathbf{e}_2}{\|\mathbf{u}\|^2}\mathbf{u} = u_2\mathbf{u}$$

$$A = \begin{bmatrix} u_1\mathbf{u} & u_2\mathbf{u} \end{bmatrix} = \mathbf{u}\mathbf{u}^T$$





Example: $\mathbf{u} = [\cos 30^\circ \;\; \sin 30^\circ]^{\mathrm{T}}$

Projections

Projection matrix $A = \begin{bmatrix} u_1\mathbf{u} & u_2\mathbf{u} \end{bmatrix} = \mathbf{u}\mathbf{u}^T$

Columns of A linearly dependent $\Rightarrow$ rank one
Map reduces dimensionality $\Rightarrow$ area after map is zero

Projection matrix is idempotent: $A = AA$
Geometrically: once a vector projected onto a line, application of same projection leaves result unchanged

Example: $\mathbf{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ then $A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

Projections

Action of projection matrix on a vector $\mathbf{x}$:

$$A\mathbf{x} = \mathbf{u}\mathbf{u}^{\mathrm{T}}\mathbf{x} = (\mathbf{u} \cdot \mathbf{x})\mathbf{u}$$

Same result as orthogonal projections in Chapter 2

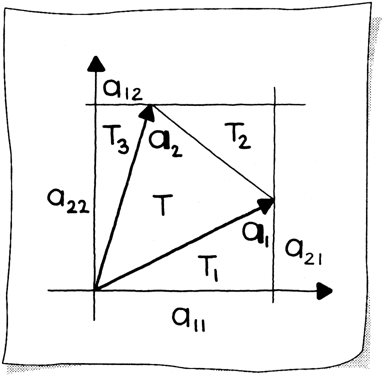
Let $\mathbf{y}$ be projection of $\mathbf{x}$ onto $\mathbf{u}$ then $\mathbf{x} = \mathbf{y} + \mathbf{y}^{\perp}$

$$A\mathbf{x} = \mathbf{u}\mathbf{u}^{\mathrm{T}}\mathbf{y} + \mathbf{u}\mathbf{u}^{\mathrm{T}}\mathbf{y}^{\perp}$$

Since $\mathbf{u}^{\mathrm{T}}\mathbf{y} = \|\mathbf{y}\|$ and $\mathbf{u}^{\mathrm{T}}\mathbf{y}^{\perp} = 0$

$$A\mathbf{x} = \|\mathbf{y}\|\mathbf{u}$$

Areas and Linear Maps: Determinants

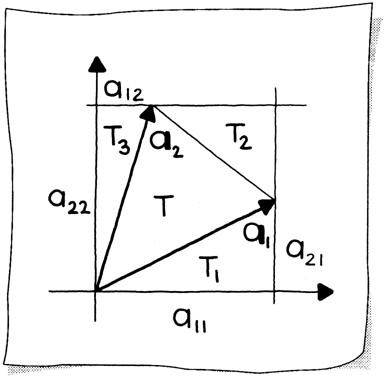


2D linear map takes
$[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{a}_1, \mathbf{a}_2]$
How does linear map affect area?

$\mathrm{area}(\mathbf{e}_1, \mathbf{e}_2) = 1$
(Square spanned by $[\mathbf{e}_1, \mathbf{e}_2]$)

P = area of parallelogram spanned by $[\mathbf{a}_1, \mathbf{a}_2]$
$P = 2T$

Areas and Linear Maps: Determinants



Area T of triangle formed by $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$T = a_{1,1}a_{2,2} - T_1 - T_2 - T_3$$

Observe that

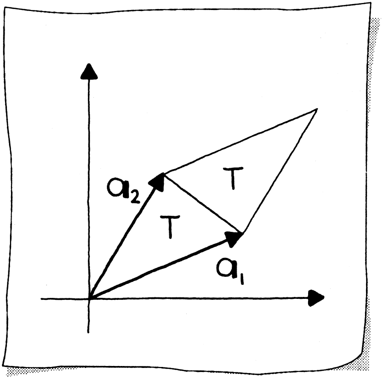
$$T_1 = \frac{1}{2}a_{1,1}a_{2,1}$$

$$T_2 = \frac{1}{2}(a_{1,1} - a_{1,2})(a_{2,2} - a_{2,1})$$

$$T_3 = \frac{1}{2}a_{1,2}a_{2,2}$$

$$T = \frac{1}{2}a_{1,1}a_{2,2} - \frac{1}{2}a_{1,2}a_{2,1}$$

$$P = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$



P: (signed) area of the parallelogram spanned by $A = [\mathbf{a}_1, \mathbf{a}_2]$
This is the determinant of A
— Notation: $\det A$ or $|A|$

$$|A| = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

Areas and Linear Maps: Determinants

Determinant characterizes a linear map:

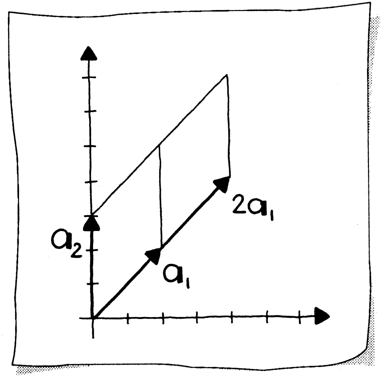
- If $|A| = 1$ then linear map does not change areas
- If $0 \leq |A| < 1$ then linear map shrinks areas
- If $|A| = 0$ then matrix is rank deficient
- If $|A| > 1$ then linear map expands areas
- If $|A| < 0$ then linear map changes the orientation of objects

$$\begin{vmatrix} 1 & 5 \\ 0 & 1 \end{vmatrix} = (1)(1) - (5)(0) = 1$$

$$\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = (1)(-1) - (0)(0) = -1$$

$$\begin{vmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{vmatrix} = (0.5)(0.5) - (0.5)(0.5) = 0$$

Areas and Linear Maps: Determinants

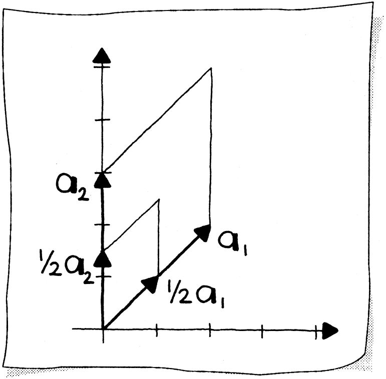


$$|c\mathbf{a}_1, \mathbf{a}_2| = c|\mathbf{a}_1, \mathbf{a}_2| = c|A|$$

If one column of A scaled by c
then A's determinant scaled by c

Sketch: $c = 2$

Areas and Linear Maps: Determinants

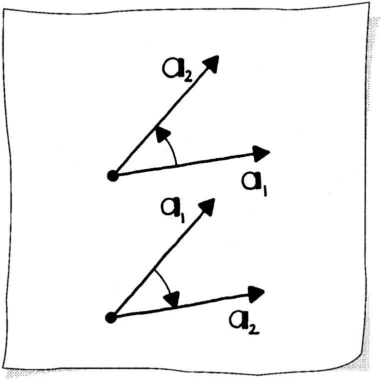


$$|c\mathbf{a}_1, c\mathbf{a}_2| = c^2|\mathbf{a}_1, \mathbf{a}_2| = c^2|A|$$

If both columns of A scaled by c
then A's determinant scaled by c^2

Sketch: $c = 1/2$

Areas and Linear Maps: Determinants



Two 2D vectors whose determinant is positive: right-handed
Standard example: $\mathbf{e}_1$ and $\mathbf{e}_2$

Two 2D vectors whose determinant is negative are called left-handed

Area sign change when columns interchanged: $|\mathbf{a}_1, \mathbf{a}_2| = -|\mathbf{a}_2, \mathbf{a}_1|$
Verified using the definition of a determinant:

$$|\mathbf{a}_2, \mathbf{a}_1| = a_{1,2}a_{2,1} - a_{2,2}a_{1,1}$$

Composing Linear Maps

Matrix product used to compose linear maps:

$$\mathbf{v}' = A\mathbf{v}$$

$$\mathbf{v}'' = B\mathbf{v}' = B(A\mathbf{v}) = BA\mathbf{v} = C\mathbf{v}$$

$$C = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} b_{1,1}a_{1,1} + b_{1,2}a_{2,1} & b_{1,1}a_{1,2} + b_{1,2}a_{2,2} \\ b_{2,1}a_{1,1} + b_{2,2}a_{2,1} & b_{2,1}a_{1,2} + b_{2,2}a_{2,2} \end{bmatrix}$$

Element $c_{i,j}$ computed as dot product of B's i^{th} row and A's j^{th} column

Example:

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\mathbf{v}' A\mathbf{v} == \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

$$\mathbf{v}'' = B\mathbf{v}' = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

Compute $\mathbf{v}''$ using the matrix product BA:

$$C = BA = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 3 & -3 \end{bmatrix}$$

Verify that $\mathbf{v}'' = C\mathbf{v}$

Composing Linear Maps

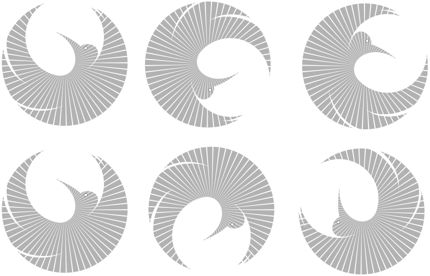
Neat way to arrange two matrices when forming their product

$$
\begin{array}{cc|cc}
 & & -\mathbf{1} & 2 \\
 & & \mathbf{0} & 3 \\
\hline
0 & -2 & & \\
-\mathbf{3} & \mathbf{1} & \mathbf{3} &
\end{array}
$$

$$
\begin{array}{cc|cc}
 & & -1 & 2 \\
 & & 0 & 3 \\
\hline
0 & -2 & 0 & -6 \\
-3 & 1 & 3 & -3
\end{array}
$$

Composing Linear Maps

Linear map composition is order dependent



Top: rotate by -120°, then reflect about the (rotated) $\mathbf{e}_1$-axis
Bottom: reflect, then rotate

Matrix products differs significantly from products of real numbers:
Matrix products are not *commutative*

$$AB \neq BA$$

Some maps to commute – example: 2D rotations

Composing Linear Maps

Rank of a composite map:

$$\mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$$

Matrix multiplication does not increase rank

Special composition: idempotent matrix $A = AA$ or $A = A^2$
Thus $A\mathbf{v} = AA\mathbf{v}$

More on Matrix Multiplication

Vectors as matrices: $\mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$

Example: Let $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$$\mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v} = 15$$

$$(\mathbf{u}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}} = \mathbf{v}^{\mathrm{T}}\mathbf{u}$$

$$[\mathbf{u}^{\mathrm{T}}\mathbf{v}]^{\mathrm{T}} = (\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix})^{\mathrm{T}} = [15]^{\mathrm{T}} = 15$$

$$\mathbf{v}^{\mathrm{T}}\mathbf{u} = \begin{bmatrix} -3 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [15] = 15$$

More on Matrix Multiplication

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$$

$$(AB)^{\mathrm{T}} = \left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ \mathbf{a_{2,1}} & \mathbf{a_{2,2}} \end{bmatrix} \begin{bmatrix} \mathbf{b_{1,1}} & b_{1,2} \\ \mathbf{b_{2,1}} & b_{2,2} \end{bmatrix} \right)^{\mathrm{T}} = \begin{bmatrix} c_{1,1} & \mathbf{c_{1,2}} \\ c_{2,1} & c_{2,2} \end{bmatrix}$$

$$B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix} \mathbf{b}^{\mathrm{T}}_{\mathbf{1,1}} & \mathbf{b}^{\mathrm{T}}_{\mathbf{1,2}} \\ b^{\mathrm{T}}_{2,1} & b^{\mathrm{T}}_{2,2} \end{bmatrix}) \begin{bmatrix} a^{\mathrm{T}}_{1,1} & \mathbf{a}^{\mathrm{T}}_{\mathbf{1,2}} \\ a^{\mathrm{T}}_{2,1} & \mathbf{a}^{\mathrm{T}}_{\mathbf{2,2}} \end{bmatrix} = \begin{bmatrix} c_{1,1} & \mathbf{c_{1,2}} \\ c_{2,1} & c_{2,2} \end{bmatrix}$$

Since $b_{i,j} = b^{\mathrm{T}}_{j,i}$ identical dot product calculated to form $c_{1,2}$

More on Matrix Multiplication

Determinant of a product matrix

$$|AB| = |A||B|$$

B scales objects by $|B|$ and A scales objects by $|A|$
Composition of the maps scales by the product of the individual scales
Example: two scalings

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$|A| = 1/4$ and $|B| = 16 \Rightarrow A$ scales down, and B scales up
Effect of B's scaling greater than A's
$AB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ scales up: $|AB| = |A||B| = 4$

Exponents for matrices: $A^r = \underbrace{A \cdot \ldots \cdot A}_{r \text{ times}}$

Some rules: $A^{r+s} = A^r A^s \qquad A^{rs} = (A^r)^s \qquad A^0 = I$

Matrix Arithmetic Rules

Matrix sizes must be compatible for operations to be performed

— matrix addition: matrices to have the same dimensions

— matrix multiplication: "inside" dimensions to be equal

Let A's dimensions be $m \times r$ and B's are $r \times n$

Product $C = AB$ is permissible since inside dimension r is shared

Resulting matrix C dimension $m \times n$

Commutative Law for Addition: $A + B = B + A$

Associative Law for Addition: $A + (B + C) = (A + B) + C$

No Commutative Law for Multiplication: $AB \neq BA$

Associative Law for Multiplication: $A(BC) = (AB)C$

Distributive Law: $A(B + C) = AB + AC$

Distributive Law: $(B + C)A = BA + CA$

Matrix Arithmetic Rules

Rules involving scalars:

$$a(B + C) = aB + aC$$

$$(a + b)C = aC + bC$$

$$(ab)C = a(bC)$$

$$a(BC) = (aB)C = B(aC)$$

Rules involving the transpose:

$$(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$$

$$(bA)^{\mathrm{T}} = bA^{\mathrm{T}}$$

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$$

$$A^{\mathrm{T}^{\mathrm{T}}} = A$$

WYSK

- linear combination
- matrix form
- pre-image and image
- domain and range
- column space
- identity matrix
- matrix addition
- distributive law
- transpose matrix
- symmetric matrix
- rank of a matrix
- rank deficient
- singular matrix
- linear space or vector space
- subspace
- linearity property
- scalings
- action ellipse
- reflections
- rotations
- rigid body motions
- shears
- projections
- parallel projection
- oblique projection
- dyadic matrix
- idempotent map
- determinant
- signed area
- matrix multiplication
- composite map
- non-commutative property of matrix multiplication
- transpose of a product or m of matrices
- rules of matrix arithmetic