Practical Linear Algebra: A Geometry Toolbox

Fourth Edition

Chapter 5: 2×2 Linear Systems

Gerald Farin & Dianne Hansford

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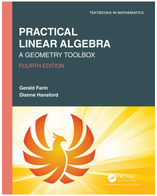
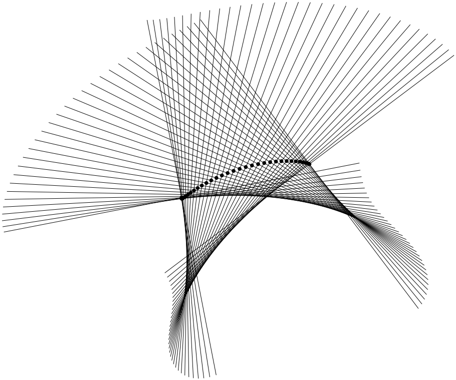


Outline

Introduction to 2 × 2 Linear Systems



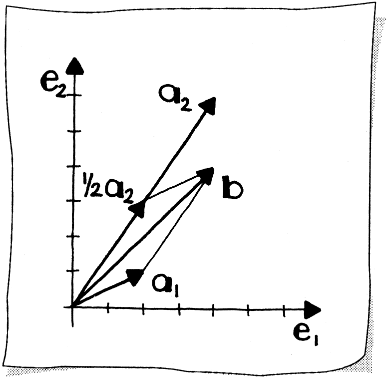
Two families of lines are shown

Intersections of corresponding line pairs marked

For each intersection:
solve a 2 × 2 linear system

Skew Target Boxes Revisited

Geometry of a 2×2 system



$\mathbf{a}_1$ and $\mathbf{a}_2$ define a skew target box

Given $\mathbf{b}$ with respect to the $[\mathbf{e}_1, \mathbf{e}_2]$-system:
What are the components of $\mathbf{b}$ with respect to the $[\mathbf{a}_1, \mathbf{a}_2]$-system?

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$1 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Skew Target Boxes Revisited

Two equations in the two unknowns u_1 and u_2

$$\begin{aligned} 2u_1 + 4u_2 &= 4 \\ u_1 + 6u_2 &= 4 \end{aligned}$$

Solution: $u_1 = 1$ and $u_2 = 1/2$

This chapter dedicated to solving these equations

The Matrix Form

Two equations

$$a_{1,1}u_1 + a_{1,2}u_2 = b_1$$
$$a_{2,1}u_1 + a_{2,2}u_2 = b_2$$

Also called a linear system

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$A\mathbf{u} = \mathbf{b}$$

$\mathbf{u}$ called the solution of linear system

Previous example:

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

The Matrix Form

Recall geometric interpretation of $A\mathbf{u} = \mathbf{b}$:
Express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$

$$u_1\mathbf{a}_1 + u_2\mathbf{a}_2 = \mathbf{b}$$

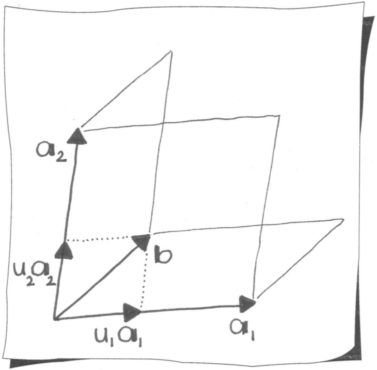
At least one solution: linear system called consistent
Otherwise: called inconsistent

Three possibilities for solution space:

1. Exactly one solution vector $\mathbf{u}$
   $|A| \neq 0$
   matrix has full rank and is non-singular
2. No solution (system is inconsistent)
3. Infinitely many solutions

(Sketches of each case to come)

A Direct Approach: Cramer's Rule



$$u_1 = \frac{\text{area}(\mathbf{b}, \mathbf{a}_2)}{\text{area}(\mathbf{a}_1, \mathbf{a}_2)} \quad u_2 = \frac{\text{area}(\mathbf{a}_1, \mathbf{b})}{\text{area}(\mathbf{a}_1, \mathbf{a}_2)}$$

Ratios of areas

Shear parallelogram formed by
— $\mathbf{b}, \mathbf{a}_2$ onto $\mathbf{a}_1$
— $\mathbf{b}, \mathbf{a}_1$ onto $\mathbf{a}_2$
(Shears preserve areas)

Signed area of a parallelogram given by determinant

A Direct Approach: Cramer's Rule

Example:

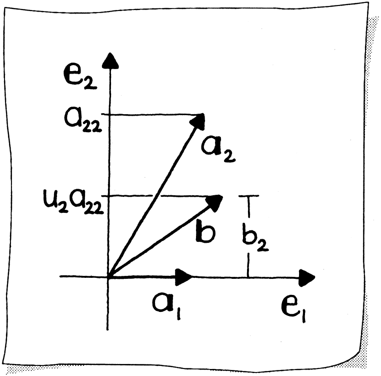
$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$u_1 = \frac{\begin{vmatrix} 4 & 4 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}} = \frac{8}{8} \qquad u_2 = \frac{\begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}} = \frac{4}{8}$$

What if area spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ is zero?

Cramer's rule primarily of theoretical importance
For larger systems: expensive and numerically unstable

Gauss Elimination



Special 2 × 2 linear system:

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ 0 & a_{2,2} \end{bmatrix} \mathbf{u} = \mathbf{b}$$

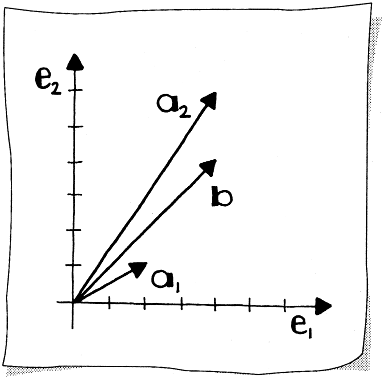
Matrix is called *upper triangular*

Solve with back substitution:

$$u_2 = b_2 / a_{2,2}$$

$$u_1 = \frac{1}{a_{1,1}}(b_1 - u_2 a_{1,2})$$

Diagonal elements key: called pivots



Any linear system with non-singular matrix may be *transformed* to upper triangular via forward elimination

Process of forward elimination followed by back substitution is called Gauss elimination

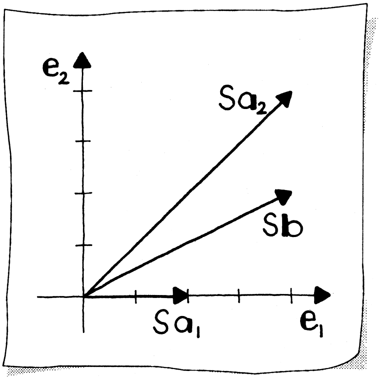
Example:

$$u_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Find u_1 and u_2

Key fact: *linear maps do not change linear combinations*

Gauss Elimination



Apply the same linear map to all vectors in system
then factors u_1 and u_2 won't change:

$$S\left(u_1\begin{bmatrix}2\\1\end{bmatrix}+u_2\begin{bmatrix}4\\6\end{bmatrix}\right)=S\begin{bmatrix}4\\4\end{bmatrix}$$

Shear parallel to the $\mathbf{e}_2$-axis so that

$$\begin{bmatrix}2\\1\end{bmatrix} \text{ is mapped to } \begin{bmatrix}2\\0\end{bmatrix}$$

$$\text{Shear matrix:} \quad S=\begin{bmatrix}1 & 0\\-1/2 & 1\end{bmatrix}$$

elementary matrix: applies one operation

Gauss Elimination

Transformed system:

$$\begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Next: back substitution

$$u_2 = 2/4 = 1/2,$$
$$u_1 = \frac{1}{2}\left(4 - 4 \times \frac{1}{2}\right) = 1$$

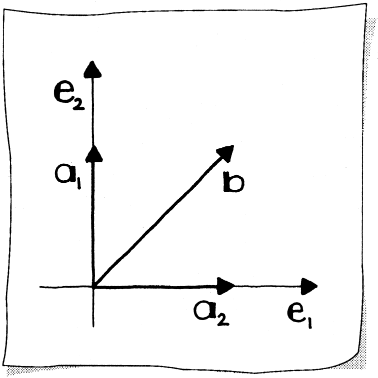
2×2 linear systems:
only one matrix entry to zero in the forward elimination procedure
(More algorithmic approach in Chapter 12 Gauss for Linear Systems)

Algebraically: transformed system by modifying the second row only

$$\text{row}_2 = -\frac{1}{2}\text{row}_1 + \text{row}_2.$$

Called an elementary row operation

Pivoting



Example:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Shearing $\mathbf{a}_1$ onto $\mathbf{e}_1$-axis will not work

Solution: exchange two equations

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Exchanging equations (rows) so pivot is the largest in absolute value called row or partial pivoting

Used to improve numerical stability

Pivoting

Row exchange is a linear map

It is represented by the permutation matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Another example of an elementary matrix resulting in an elementary row operation

Pivoting

Example:

$$\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Shear $\mathbf{a}_1$ onto the $\mathbf{e}_1$-axis

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -9999 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9998 \end{bmatrix}$$

Performing back substitution

$$\mathbf{u}_t = \begin{bmatrix} 1.0001 \\ 0.9998\bar{9} \end{bmatrix} \qquad (\text{"true" solution})$$

Suppose machine only stores three digits — system stored as

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -10000 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -10000 \end{bmatrix},$$

$$\mathbf{u}_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (\text{"round-off" solution})$$

Not very close to the true solution $\mathbf{u}_t$: $\|\mathbf{u}_t - \mathbf{u}_r\| = 1.0001$

Pivoting

Pivoting dampers effects of round-off

$$\begin{bmatrix} 1 & 1 \\ 0.0001 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Forward elimination

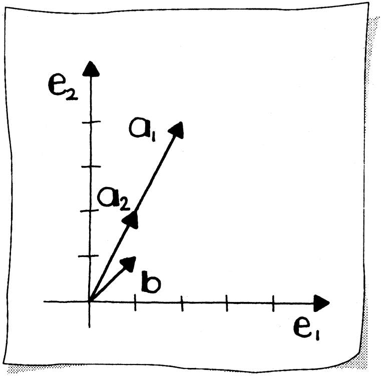
$$\begin{bmatrix} 1 & 1 \\ 0 & 0.9999 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.9998 \end{bmatrix}$$

$$\mathbf{u}_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (\text{“pivoting” solution})$$

Closer to “true” solution: $\|\mathbf{u}_t - \mathbf{u}_p\| = 0.00014$

Unsolvable Systems

$\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent



$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Forward elimination
(shear $\mathbf{a}_1$ onto $\mathbf{e}_1$-axis):

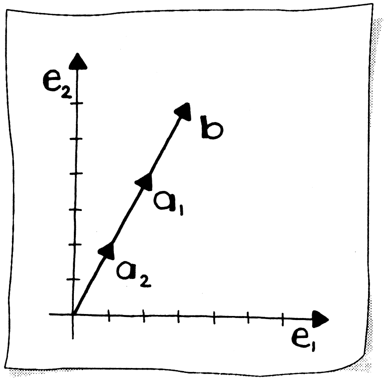
$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Last equation: $0 = -1$
System is *inconsistent* $\Rightarrow$ no solution

Approximate solution via least squares methods

Underdetermined Systems

$\mathbf{b}$ a multiple of $\mathbf{a}_1$ or $\mathbf{a}_2$



$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Forward elimination
(shear $\mathbf{a}_1$ onto $\mathbf{e}_1$-axis):

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

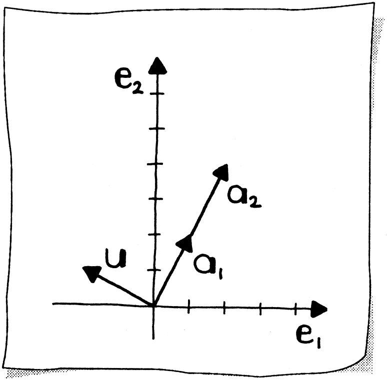
Last equation: $0 = 0$
true, but a bit trivial

- This is one equation written twice
- System is underdetermined
- System consistent: at least one solution exists

Example: set $u_2 = 1$ then $u_1 = 1$

Homogeneous Systems

$A\mathbf{u} = \mathbf{0}$
Homogeneous: Right-hand side consists of zero vector



Trivial solution: $\mathbf{u} = \mathbf{0}$
— usually of little interest

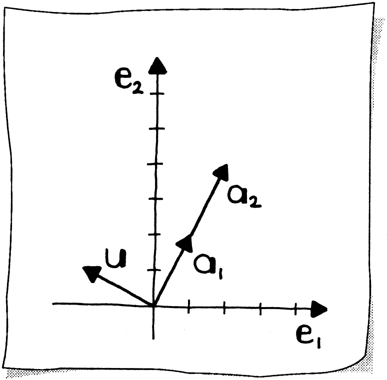
If solution $\mathbf{u} \neq \mathbf{0}$ exists
then all $c\mathbf{u}$ are solutions
$\Rightarrow$ *infinite number of solutions*

Vectors $\mathbf{u}$ that satisfy the homogeneous system are orthogonal to the row vectors

Not all homogeneous systems have a non-trivial solution
— 2×2 matrices: only rank 1 maps
$\Rightarrow$ $\mathbf{a}_1$ and $\mathbf{a}_2$ linearly dependent

If only trivial solution exist $\Rightarrow$ A invertible

Homogeneous Systems



$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbf{a}_2 = 2\mathbf{a}_1 \Rightarrow A$ maps all vectors onto line defined by $\mathbf{0}, \mathbf{a}_1$

Forward elimination:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Pick $u_2 = 1$ then
back substitution gives $u_1 = -2$

Any $c\mathbf{u}$ perpendicular to $\mathbf{a}_1$ is a solution: $\mathbf{a}_1 \cdot \mathbf{u} = 0$
— all such $\mathbf{u}$ make up kernel or null space of A

Homogeneous Systems

Example: only the trivial solution

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Columns of A are linearly independent
A does not reduce dimensionality $\Rightarrow$ cannot map $\mathbf{u} \neq \mathbf{0}$ to $\mathbf{0}$
Forward elimination:

$$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Back substitution: $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Homogeneous Systems

Example: row pivoting not helpful – need column pivoting

$$\begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} \mathbf{u} = \mathbf{0}.$$

Column pivoting:

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \mathbf{0}.$$

(Exchange unknowns too)
Set $u_1 = 1$ and back substitution results in $u_2 = 0$
Solutions: $\mathbf{u} = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Kernel

All vectors $\mathbf{u}$ that satisfy a homogeneous system

$$A\mathbf{u} = \mathbf{0}$$

make up the kernel or null space of the matrix

Vectors $\mathbf{u}$ in the kernel are orthogonal to the row space of A

The dimension of the kernel is called the nullity of A

For 2×2 matrices:

$$\text{rank} + \text{nullity} = 2$$

Kernel

Example: Homogeneous system with non-trivial solution
Rank $= 1$ Nullity $= 1$ Notice that $\mathbf{a}_2 = 2\mathbf{a}_1$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example: Homogeneous system with only trivial solution
Rank $= 2$ Nullity $= 0$ $\mathbf{a}_1$ and $\mathbf{a}_2$ linearly independent

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Kernel

Kernel = the central, most important part of something

In linear algebra, it reveals information about a linear map (matrix) and also the solution to a linear system.

The kernel is a subspace of the domain with the following properties

1. Always contains the zero vector since $A\mathbf{0} = \mathbf{0}$
2. If $\mathbf{u}$ is in the kernel, then $c\mathbf{u}$ is in the kernel: $c(A\mathbf{u}) = c\mathbf{0}$, thus $A(c\mathbf{u}) = \mathbf{0}$
3. If $\mathbf{u}$ and $\mathbf{v}$ are in the kernel, then $\mathbf{u} + \mathbf{v}$ is in the kernel (distributive law)

Kernel

Why is knowledge of the kernel useful?

Reveals the existence and uniqueness of a solution

If the nullity $= 0$, then the solution is unique.

If the nullity ≥ 1, then the solution is not unique
and the nullity reveals the number of parameters available to specify a solution

Kernel

Example: Rank 1, Nullity 1

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad \text{with specific solution} \quad \mathbf{u}_s = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Homogeneous linear system

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{with a kernel solution} \quad \mathbf{u}_k = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

Then all vectors

$$\mathbf{u} = \mathbf{u}_s + c\mathbf{u}_k$$

are solutions to the linear system since

$$A(\mathbf{u}_s + c\mathbf{u}_k) = A\mathbf{u}_s + Ac\mathbf{u}_k = A\mathbf{u}_s$$

$\Rightarrow$ there is a one parameter family of solutions

Check an example with $c = 2$

Undoing Maps: Inverse Matrices

How to *undo* a linear map

Given $A\mathbf{u} = \mathbf{b}$
What matrix B maps $\mathbf{b}$ back to $\mathbf{u}$: $\mathbf{u} = B\mathbf{b}$?
B is the inverse map

Recall: shears can be used to zero matrix elements

$$S_1 A\mathbf{u} = S_1\mathbf{b}.$$

Example:

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Undoing Maps: Inverse Matrices

Second shear: $S_2 S_1 A\mathbf{u} = S_2 S_1 \mathbf{b}$ (map new $\mathbf{a}_2$ to the $\mathbf{e}_2$-axis)

$$S_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{results in} \quad \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Non-uniform scaling S_3 (map $\mathbf{a}_1$, $\mathbf{a}_2$ onto $\mathbf{e}_1$, $\mathbf{e}_2$)

$$S_3 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} \quad \text{results in} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

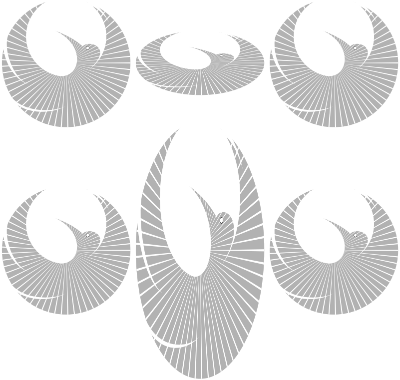
All together:

$$S_3 S_2 S_1 A\mathbf{u} = S_3 S_2 S_1 \mathbf{b}$$
$$I\mathbf{u} = A^{-1}\mathbf{b}$$

I called the identity matrix
A^{-1} called the inverse matrix and A called invertible

Undoing Maps: Inverse Matrices

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I \qquad\qquad I^{-1} = I$$



Inverse of a scaling:

$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}^{-1} = \begin{bmatrix} 1/s & 0 \\ 0 & 1/t \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Top: original Phoenix, scale, inverse scale
Bottom: original Phoenix, inverse scale, original scale

Undoing Maps: Inverse Matrices

Top: original Phoenix, shear, inverse shear
Bottom: original Phoenix, inverse shear, original shear

Undoing Maps: Inverse Matrices

Rotation matrices:

$$R_{-\alpha} = R_\alpha^{-1} = R_\alpha^{\mathrm{T}}$$

Rotation matrix is an orthogonal matrix:

$$A^{-1} = A^{\mathrm{T}}$$

Column vectors satisfy $\|\mathbf{a}_1\| = 1$, $\|\mathbf{a}_2\| = 1$ and $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$

$\Rightarrow$ vectors called orthonormal

$\Rightarrow$ these linear maps called *rigid body motions*

Characterized by determinant $= \pm 1$

Undoing Maps: Inverse Matrices

$$A^{-1^{-1}} = A \qquad \text{and} \qquad A^{-1^{\mathrm{T}}} = A^{\mathrm{T}^{-1}}$$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \end{bmatrix}$$

Top: I, A^{-1}, $A^{-1^{\mathrm{T}}}$

Bottom: I, A^{T}, $A^{\mathrm{T}^{-1}}$

Undoing Maps: Inverse Matrices

How to compute A's inverse?
Start with

$$AA^{-1} = I$$

Denote two (unknown) columns of A^{-1} by $\bar{\mathbf{a}}_1$ and $\bar{\mathbf{a}}_2$
Denote columns of I by $\mathbf{e}_1$ and $\mathbf{e}_2$

$$A \begin{bmatrix} \bar{\mathbf{a}}_1 & \bar{\mathbf{a}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$$

Short for two linear systems

$$A\bar{\mathbf{a}}_1 = \mathbf{e}_1 \quad \text{and} \quad A\bar{\mathbf{a}}_2 = \mathbf{e}_2$$

Both systems have the same matrix A

Defining a Map

Matrices map vectors to vectors

If $\mathbf{v}_1$ and $\mathbf{v}_2$ mapped to $\mathbf{v}'_1$ and $\mathbf{v}'_2$, what matrix A did it?

$$A\mathbf{v}_1 = \mathbf{v}'_1 \quad \text{and} \quad A\mathbf{v}_2 = \mathbf{v}'_2$$

Combining into a matrix equation:

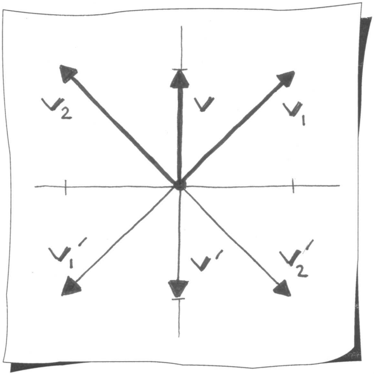
$$A[\mathbf{v}_1 \quad \mathbf{v}_2] = [\mathbf{v}'_1 \quad \mathbf{v}'_2] \qquad \text{or} \qquad AV = V'$$

Solution: find V^{-1}, then $A = V'V^{-1}$

$\mathbf{v}_1$ and $\mathbf{v}_2$ must be linearly independent for V^{-1} to exist

Defining a Map

Example:



$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1' = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

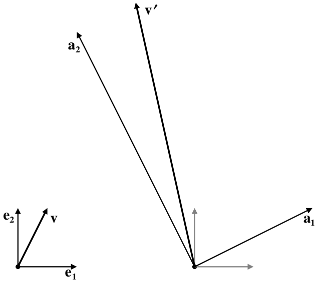
$$V^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$A = V'V^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Change of Basis

Basics of coordinate systems

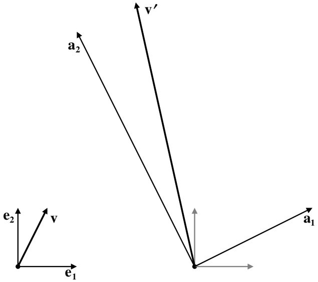


$$\text{Figure (left):} \qquad \mathbf{v} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \frac{1}{2}\mathbf{e}_1 + 1\mathbf{e}_2$$

Basis vectors and the origin establish a grid for navigating 2D space

Change of Basis

Choose any set of linearly independent vectors
Figure (right): $[\mathbf{a}_1, \mathbf{a}_2]$-system



$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \quad \mathbf{v}' = \frac{1}{2}\mathbf{a}_1 + \mathbf{a}_2$$

In the $[\mathbf{a}_1, \mathbf{a}_2]$-system

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{defined as} \quad \mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 \quad \text{and} \quad \mathbf{a}_2 = 0\mathbf{a}_1 + 1\mathbf{a}_2$$

Change of Basis

$\mathbf{a}_1$ and $\mathbf{a}_2$ play the same role in the $[\mathbf{a}_1, \mathbf{a}_2]$-system as do $\mathbf{e}_1$ and $\mathbf{e}_2$ in the $[\mathbf{e}_1, \mathbf{e}_2]$-system

Basis vector must not be orthogonal nor unit length but sometimes desirable

Change of Basis

Change of basis problems

1. Given $\mathbf{v}'_{\mathbf{a}} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}_{\mathbf{a}}$ in the $[\mathbf{a}_1, \mathbf{a}_2]$-system, what are the components of this vector in the $[\mathbf{e}_1, \mathbf{e}_2]$-system, referred to as $\mathbf{v}'_{\mathbf{e}}$?
2. Given $\mathbf{v}_{\mathbf{e}} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}_{\mathbf{e}}$ in the $[\mathbf{e}_1, \mathbf{e}_2]$-system, what are the components of this vector in the $[\mathbf{a}_1, \mathbf{a}_2]$-system, referred to as $\mathbf{v}_{\mathbf{a}}$?

– Subscripts have been added to the vectors to make clear their defining basis vectors.
– Extra square brackets are added when needed to improve readability.

Change of Basis

Question 1:

Write $[\mathbf{a}_1, \mathbf{a}_2]$-system vectors in terms of the $[\mathbf{e}_1, \mathbf{e}_2]$-system vectors

$$[\mathbf{a}_1]_\mathbf{e} = 2\mathbf{e}_1 + 1\mathbf{e}_2 \quad \text{and} \quad [\mathbf{a}_2]_\mathbf{e} = -2\mathbf{e}_1 + 4\mathbf{e}_2$$

Vector $\mathbf{v}'$ with respect to the $[\mathbf{e}_1, \mathbf{e}_2]$-system

$$\mathbf{v}'_\mathbf{e} = \frac{1}{2}[\mathbf{a}_1]_\mathbf{e} + 1[\mathbf{a}_2]_\mathbf{e}$$

In matrix form

$$\mathbf{v}'_\mathbf{e} = A \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}_\mathbf{a} = \begin{bmatrix} -1 \\ 9/2 \end{bmatrix}_\mathbf{e} \quad \text{where} \quad A = \begin{bmatrix} [\mathbf{a}_1]_\mathbf{e} & [\mathbf{a}_2]_\mathbf{e} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}$$

A maps a vector in the $[\mathbf{a}_1, \mathbf{a}_2]$-system to one in the $[\mathbf{e}_1, \mathbf{e}_2]$-system
Called a change of basis matrix

Question 2:

Apply the inverse map to the matrix in Question 1

$$A^{-1}\begin{bmatrix}-1\\9/2\end{bmatrix}_{\mathbf{e}} = \begin{bmatrix}1/2\\1\end{bmatrix}_{\mathbf{a}} \qquad A^{-1} = \begin{bmatrix}2/5 & 1/5\\-1/10 & 1/5\end{bmatrix}$$

Components of $\mathbf{e}_1$ and $\mathbf{e}_2$ in the $[\mathbf{a}_1, \mathbf{a}_2]$-system are revealed in A^{-1}

$$\mathbf{e}_1 = \frac{2}{5}\mathbf{a}_1 - \frac{1}{10}\mathbf{a}_2 \quad \text{and} \quad \mathbf{e}_2 = \frac{1}{5}\mathbf{a}_1 + \frac{1}{5}\mathbf{a}_2.$$

We could have constructed A^{-1} by solving two 2×2 linear systems

$$A[\mathbf{e}_1]_{\mathbf{a}} = \mathbf{e}_1 \quad \text{and} \quad A[\mathbf{e}_2]_{\mathbf{a}} = \mathbf{e}_2 \quad \text{then} \quad A^{-1} = [[\mathbf{e}_1]_{\mathbf{a}} \;\; [\mathbf{e}_2]_{\mathbf{a}}]$$

The answer to Question 2:

$$\mathbf{v}_{\mathbf{a}} = A^{-1}\mathbf{v}_{\mathbf{e}} = \begin{bmatrix}4/10\\3/20\end{bmatrix}$$

Change of Basis

Application: coordinate system axes rotation

Given: $\mathbf{u_a}$

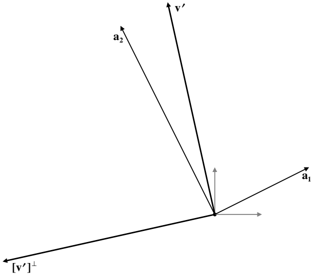
Find: the matrix for a 90° in the $[\mathbf{a}_1, \mathbf{a}_2]$-system

Solution:

1. construct change of basis matrix A: map $\mathbf{u_a}$ into the $[\mathbf{e}_1, \mathbf{e}_2]$-system
2. apply the "usual" rotation R
3. transform back to the $[\mathbf{a}_1, \mathbf{a}_2]$-system with A^{-1}

$$\mathbf{u}'_{\mathbf{a}} = A^{-1} R A \mathbf{u_a}$$

Application: coordinate system axes rotation



90° rotation of $\mathbf{v}'_\mathbf{a}$ into $[\mathbf{v}'_\mathbf{a}]^\perp$

Example:

Rotate $\mathbf{v}'_\mathbf{a}$ by 90° in the $[\mathbf{a}_1, \mathbf{a}_2]$-system

$$R' = A^{-1}RA = \begin{bmatrix} 0 & -2 \\ 1/2 & 0 \end{bmatrix}$$

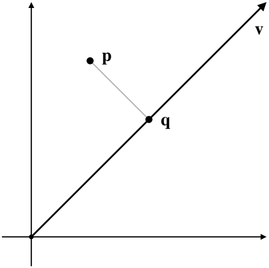
$$[\mathbf{v}'_\mathbf{a}]^\perp = R'\mathbf{v}'_\mathbf{a} = \begin{bmatrix} -2 \\ 1/4 \end{bmatrix}$$

R and R' describe the same linear map, but with respect to different bases

They are similar matrices

Change of Basis

Application: projecting a point onto a line



Let **v** form angle θ with $\mathbf{e}_1$

$$M = R_\theta P R_{-\theta}$$

where

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

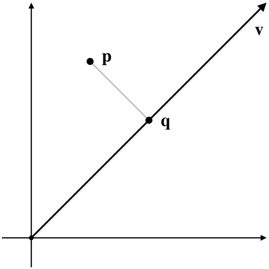
$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Project point **p** onto line defined by **v** resulting in **q** closest to **p**

P and M are similar matrices

Change of Basis

Application: projecting a point onto a line



v is not drawn normalized

Example:

$$\theta = 45^\circ \quad \mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

Projection of **p** onto **v**:

$$\mathbf{q} = M\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Application: Intersecting Lines

Two interpretations of of a linear system

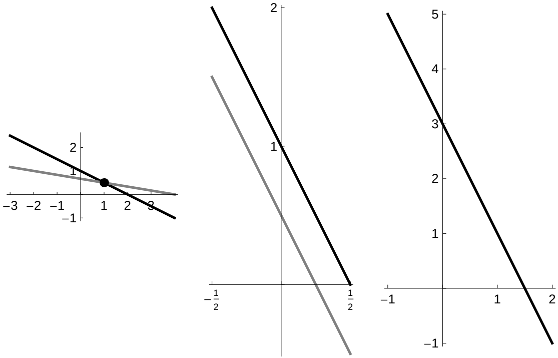
1. "column view": coordinate system or linear combination approach
2. "row view": focus on the row equations

Line intersection problems provide examples of both:
— parametric/parametric line ⇒ column view
— implicit/implicit line ⇒ row view

Choose the view that best suits given information

Application: Intersecting Lines

Linear systems from this chapter interpreted as line intersection



Left to right: unique solution, inconsistent, underdetermined

WYSK

- linear system
- solution spaces
- consistent linear system
- Cramer's rule
- upper triangular
- Gauss elimination
- forward elimination
- back substitution
- linear combination
- inverse matrix
- orthogonal matrix
- orthonormal
- rigid body motion
- inconsistent system of equations
- underdetermined system of equations
- homogeneous system
- kernel
- null space
- row pivoting
- column pivoting
- complete pivoting
- change of basis
- column and row views of linear systems