Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 8: 3D Geometry

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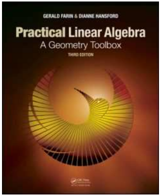
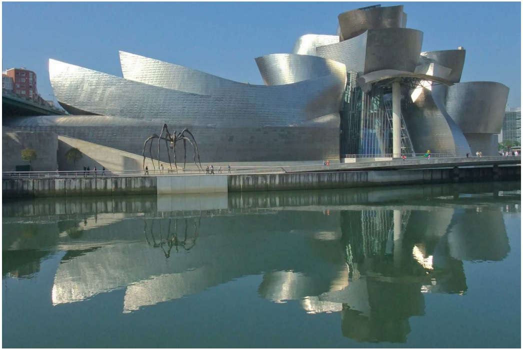
www.farinhansford.com/books/pla



Outline

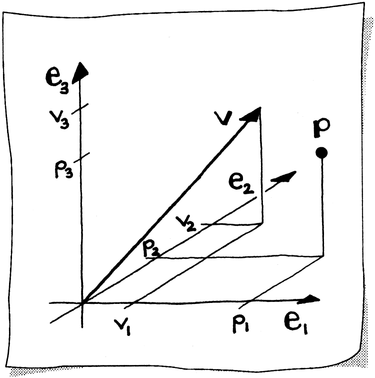
Introduction to 3D Geometry

With 3D geometry concepts we can create and analyze 3D objects
Guggenheim Museum in Bilbao, Spain. Designed by Frank Gehry



Introduction to essential building blocks of 3D geometry
— Extend 2D tools
— Encounter concepts without 2D counterparts

From 2D to 3D



$[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-coordinate system

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

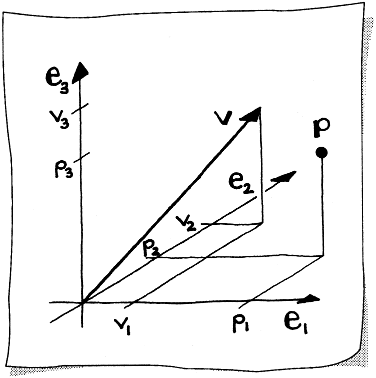
Vector in 3D: $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Components of **v** indicate displacement along each axis

v lives in 3D space $\mathbb{R}^3$

shorter: $\mathbf{v} \in \mathbb{R}^3$

From 2D to 3D



Point $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

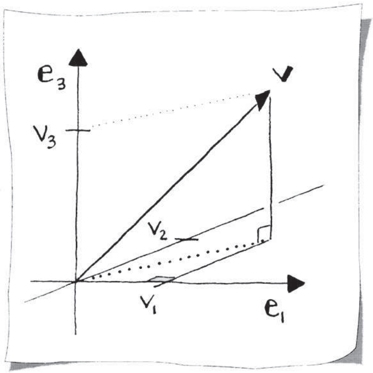
Coordinates indicate the point's location in $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system

$\mathbf{p}$ lives in Euclidean 3D-space $\mathbb{E}^3$

shorter: $\mathbf{p} \in \mathbb{E}^3$

From 2D to 3D

Basic 3D vector properties



3D *zero vector:* $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Sketch shows components $\mathbf{v}$

Notice the two right triangles

Apply *Pythagorean theorem* twice

length or *Euclidean norm* of $\mathbf{v}$

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Interpret as distance, speed, or force

Scaling by k: $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

Normalized vector: $\|\mathbf{v}\| = 1$.

Example:

Normalize the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Calculate $\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ Normalized vector $\mathbf{w}$ is

$$\mathbf{w} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \approx \begin{bmatrix} 0.27 \\ 0.53 \\ 0.80 \end{bmatrix}$$

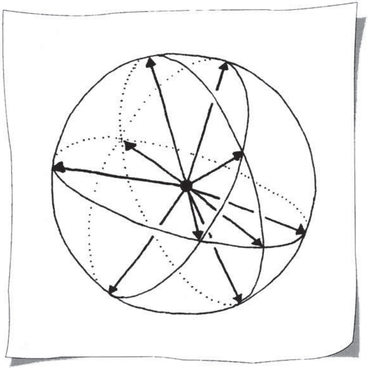
Check that $\|\mathbf{w}\| = 1$

Scale $\mathbf{v}$ by $k = 2$:

$$2\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \qquad \|2\mathbf{v}\| = \sqrt{2^2 + 4^2 + 6^2} = 2\sqrt{14}$$

Verified that $\|2\mathbf{v}\| = 2\|\mathbf{v}\|$

From 2D to 3D



There are infinitely many 3D unit vectors
Sketch is a *sphere* of radius one

All the rules for combining points and vectors in 2D carry over to 3D

Dot product:

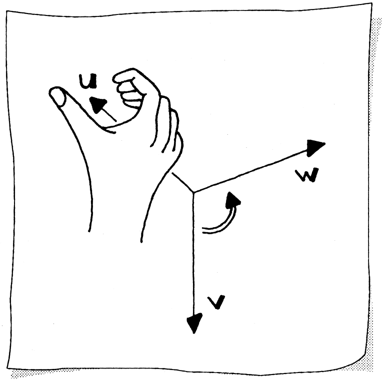
$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Cosine of the angle θ between two vectors:

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

Cross Product

Multiplication for two vectors



Dot product reveals angle between vectors

Cross product reveals orientation in $\mathbb{R}^3$

— Two vectors define a plane

— Cross product defines a 3rd vector to complete a 3D coordinate system *embedded* in the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system

$$\mathbf{u} = \mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} v_2 w_3 - w_2 v_3 \\ v_3 w_1 - w_3 v_1 \\ v_1 w_2 - w_1 v_2 \end{bmatrix}$$

Cross Product

$$\mathbf{u} = \mathbf{v} \wedge \mathbf{w}$$

Produces vector $\mathbf{u}$ that satisfies:

1. $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$

$$\mathbf{u} \cdot \mathbf{v} = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{w} = 0$$

2. Orientation of $\mathbf{u}$ follows the *right-hand rule* (see Sketch)
3. Magnitude of $\mathbf{u}$ is area of parallelogram defined by $\mathbf{v}$ and $\mathbf{w}$

Cross product produces a vector — also called a *vector product*

$\mathbf{v}$ and $\mathbf{w}$ orthogonal and unit length $\Rightarrow$ *orthonormal* $\mathbf{u}, \mathbf{v}, \mathbf{w}$

Cross Product

$$\mathbf{u} = \mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} v_2 w_3 - w_2 v_3 \\ v_3 w_1 - w_3 v_1 \\ v_1 w_2 - w_1 v_2 \end{bmatrix}$$

Each component is a 2×2 determinant
For the i^{th} component, omit the i^{th} component of $\mathbf{v}$ and $\mathbf{w}$ and negate the middle determinant:

$$\mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} \\ -\begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} \\ \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \end{bmatrix}$$

Cross Product

Example:

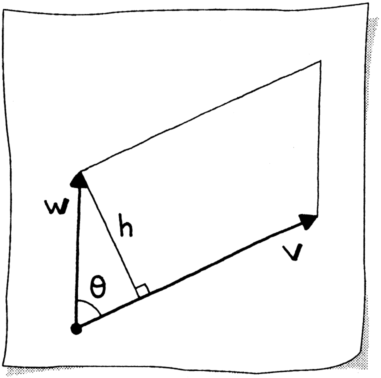
Compute the cross product of

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} 0 \times 4 - 3 \times 2 \\ 2 \times 0 - 4 \times 1 \\ 1 \times 3 - 0 \times 0 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ 3 \end{bmatrix}$$

Area of a parallelogram defined by two vectors

$$P = \|\mathbf{v} \wedge \mathbf{w}\|$$



(Analogous to 2D)
P also defined by measuring a height and side length of the parallelogram

Height $h = \|\mathbf{w}\| \sin\theta$

Side length is $\|\mathbf{v}\|$

Resulting in

$$P = \|\mathbf{v}\| \|\mathbf{w}\| \sin\theta$$

Equating two expressions:

$$\|\mathbf{v} \wedge \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin\theta$$

Example:

Compute the area of the parallelogram formed by

$$\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ then area $P = \|\mathbf{v} \wedge \mathbf{w}\| = 2\sqrt{2}$

(Verify: parallelogram is a rectangle $\Rightarrow$ area is product of edge lengths)

Also: $P = 2\sqrt{2} \sin 90^\circ = 2\sqrt{2}$

Cross Product

Lagrange's identity

Start with $\|\mathbf{v} \wedge \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$
Square both sides

$$\begin{aligned}
\|\mathbf{v} \wedge \mathbf{w}\|^2 &= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \sin^2\theta \\
&= \|\mathbf{v}\|^2\|\mathbf{w}\|^2(1 - \cos^2\theta) \\
&= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \cos^2\theta \\
&= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2
\end{aligned}$$

An expression for the area of a parallelogram in terms of a dot product

Cross Product

Properties

- Parallel vectors result in the zero vector: $\mathbf{v} \wedge c\mathbf{v} = \mathbf{0}$
- Homogeneous: $c\mathbf{v} \wedge \mathbf{w} = c(\mathbf{v} \wedge \mathbf{w})$
- Anti-symmetric: $\mathbf{v} \wedge \mathbf{w} = -(\mathbf{w} \wedge \mathbf{v})$
- Non-associative: $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) \neq (\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$, in general
- Distributive: $\mathbf{u} \wedge (\mathbf{v} + \mathbf{w}) = \mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{w}$
- Right-hand rule:
  $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \wedge \mathbf{e}_1 = \mathbf{e}_2$
- Orthogonality:

$$\begin{array}{lll} \mathbf{v} \cdot (\mathbf{v} \wedge \mathbf{w}) = 0 & \mathbf{v} \wedge \mathbf{w} \quad \text{is orthogonal to} & \mathbf{v} \\ \mathbf{w} \cdot (\mathbf{v} \wedge \mathbf{w}) = 0 & \mathbf{v} \wedge \mathbf{w} \quad \text{is orthogonal to} & \mathbf{w} \end{array}$$

Cross Product

Example: test these properties of the cross product

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Parallel vectors:

$$\mathbf{v} \wedge 3\mathbf{v} = \begin{bmatrix} 0 \times 0 - 0 \times 0 \\ 0 \times 6 - 0 \times 2 \\ 2 \times 0 - 6 \times 0 \end{bmatrix} = \mathbf{0}$$

Homogeneous:

$$4\mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} 0 \times 0 - 3 \times 0 \\ 0 \times 0 - 0 \times 8 \\ 8 \times 3 - 0 \times 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix}$$

$$4(\mathbf{v} \wedge \mathbf{w}) = 4 \begin{bmatrix} 0 \times 0 - 3 \times 0 \\ 0 \times 0 - 0 \times 2 \\ 2 \times 3 - 0 \times 0 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix}$$

Cross Product

Anti-symmetric:

$$\mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \quad \text{and} \quad -(\mathbf{w} \wedge \mathbf{v}) = -\left(\begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} \right)$$

Non-associative:

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = \begin{bmatrix} 1 \times 6 - 0 \times 1 \\ 1 \times 0 - 6 \times 1 \\ 1 \times 0 - 0 \times 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 0 \end{bmatrix}$$

which is not the same as

$$(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Cross Product

Distributive:

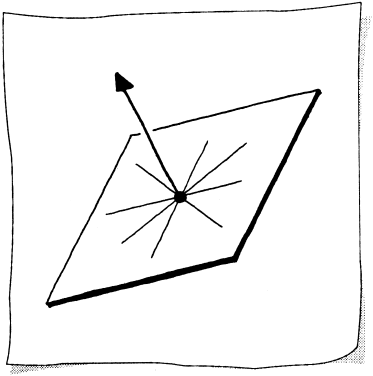
$$\mathbf{u} \wedge (\mathbf{v} + \mathbf{w}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

which is equal to

$$(\mathbf{u} \wedge \mathbf{v}) + (\mathbf{u} \wedge \mathbf{w}) = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Lines

Specifying a line with 3D geometry differs a bit from 2D



Restricted to specifying

- two points or
- a point and a vector parallel to the line

The 2D geometry item

- a point and a vector perpendicular to the line

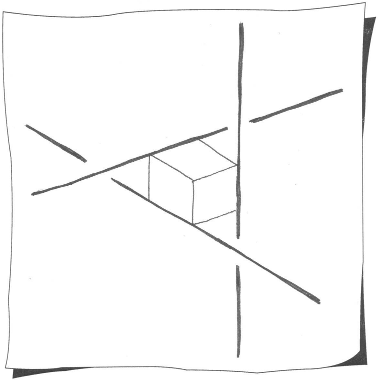
no longer works

An entire family of lines satisfies this — this family lies in a plane

$\Rightarrow$ concept of a *normal* to a 3D line does not exist

$\Rightarrow$ no 3D implicit form

Lines



Parametric form of a 3D line

$$\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$$

where $\mathbf{p} \in \mathbb{E}^3$ and $\mathbf{v} \in \mathbb{R}^3$
— same 2D line except 3D info

Points generated on line as parameter t varies

2D: two lines either intersect or they are parallel
3D: third possibility — lines are *skew*

Lines

Intersection of two lines given in parametric form

$$\mathbf{l}_1 : \quad \mathbf{l}_1(t) = \mathbf{p} + t\mathbf{v}$$
$$\mathbf{l}_2 : \quad \mathbf{l}_2(s) = \mathbf{q} + s\mathbf{w}$$

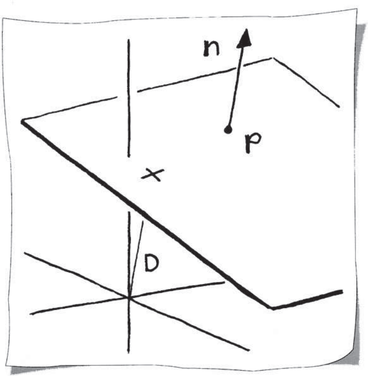
where $\mathbf{p}, \mathbf{q} \in \mathbb{E}^3$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$
Solve for t or s
Linear system

$$\hat{t}\mathbf{v} - \hat{s}\mathbf{w} = \mathbf{q} - \mathbf{p}$$

Three equations and two unknowns — *overdetermined* system
No solution exists when the lines are skew
Can find a best approximation – the least squares solution
— topic of Chapter 12

Still have concepts of perpendicular and parallel lines in 3D

Point normal plane equation



Given information:
point $\mathbf{p}$ and vector $\mathbf{n}$ bound to $\mathbf{p}$

Implicit form of a plane:
Locus of all points $\mathbf{x}$ that satisfy

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

if $\|\mathbf{n}\| = 1$
$\mathbf{n}$ called the normal to the plane

Planes

Expanding $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

$$n_1 x_1 + n_2 x_2 + n_3 x_3 - (n_1 p_1 + n_2 p_2 + n_3 p_3) = 0$$

Typically written as $Ax_1 + Bx_2 + Cx_3 + D = 0$ where

$$A = n_1 \quad B = n_2 \quad C = n_3 \quad D = -(n_1 p_1 + n_2 p_2 + n_3 p_3)$$

Example: Find implicit form of plane through the point

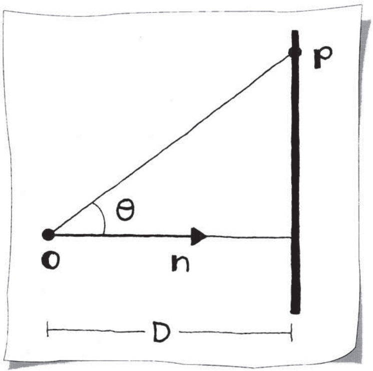
$\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ that is perpendicular to vector $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Compute $D = -(1 \times 4 + 1 \times 0 + 1 \times 0) = -4$

Plane equation is $x_1 + x_2 + x_3 - 4 = 0$

Planes

Origin to plane distance D



If coefficients A, B, C correspond to the normal to the plane
then $|D|$ describes the distance of the plane to the origin
— *perpendicular distance*

Equate

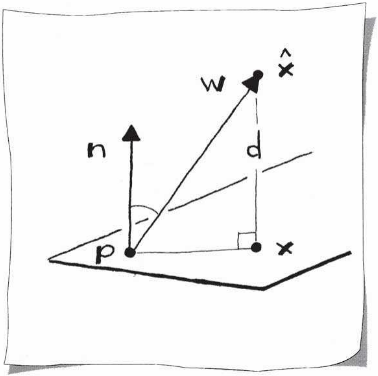
$$\cos(\theta) = \frac{D}{\|\mathbf{p}\|} \quad \text{and} \quad \cos(\theta) = \frac{\mathbf{n} \cdot \mathbf{p}}{\|\mathbf{n}\| \|\mathbf{p}\|}$$

Since normal is unit length:

$$D = \mathbf{n} \cdot \mathbf{p}$$

Planes

Point $\hat{\mathbf{x}}$ to plane distance d



$$d = A\hat{x}_1 + B\hat{x}_2 + C\hat{x}_3 + D$$

Convert a plane to point normal form:
Normalize $\mathbf{n}$ and divide the implicit equation by this factor

$$\frac{\mathbf{n} \cdot (\mathbf{x} - \mathbf{p})}{\|\mathbf{n}\|} = \frac{\mathbf{n} \cdot \mathbf{x}}{\|\mathbf{n}\|} - \frac{\mathbf{n} \cdot \mathbf{p}}{\|\mathbf{n}\|} = 0$$

Resulting in
$A' = \frac{A}{\|\mathbf{n}\|}, B' = \frac{B}{\|\mathbf{n}\|}, C' = \frac{C}{\|\mathbf{n}\|}, D' = \frac{D}{\|\mathbf{n}\|}$

Planes

Example: Plane $x_1 + x_2 + x_3 - 4 = 0$

Not in point normal form: $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\|\mathbf{n}\| = 1/\sqrt{3}$

New coefficients of the plane equation:

$$A' = B' = C' = \frac{1}{\sqrt{3}} \qquad D' = \frac{-4}{\sqrt{3}}$$

Resulting in point normal plane equation

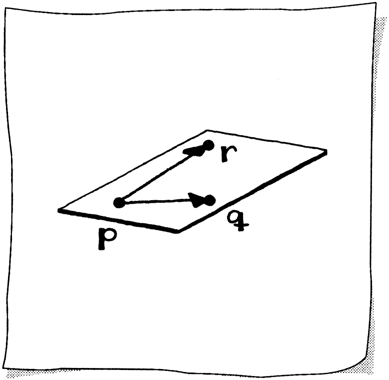
$$\frac{1}{\sqrt{3}}x_1 + \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3 - \frac{4}{\sqrt{3}} = 0$$

Distance d of the point $\mathbf{q} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ from the plane:

$$d = \frac{1}{\sqrt{3}} \times 4 + \frac{1}{\sqrt{3}} \times 4 + \frac{1}{\sqrt{3}} \times 4 - \frac{4}{\sqrt{3}} = \frac{8}{\sqrt{3}} \approx 4.6$$

$d > 0 \Rightarrow \mathbf{q}$ is on same side of plane as normal direction

Parametric plane



Given:

- three points, or
- a point and two vectors

If start with points $\mathbf{p}, \mathbf{q}, \mathbf{r}$, then form

$$\mathbf{v} = \mathbf{q} - \mathbf{p} \quad \text{and} \quad \mathbf{w} = \mathbf{r} - \mathbf{p}$$

$$\mathbf{P}(s, t) = \mathbf{p} + s\mathbf{v} + t\mathbf{w}$$

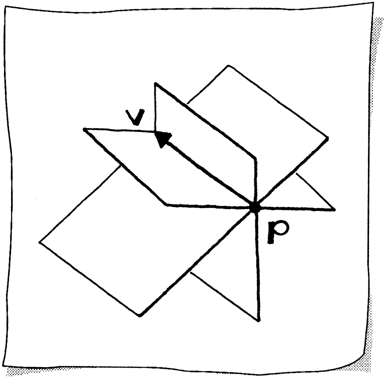
In terms of *barycentric coordinates*

$$\begin{aligned} \mathbf{P}(s, t) &= \mathbf{p} + s(\mathbf{q} - \mathbf{p}) + t(\mathbf{r} - \mathbf{p}) \\ &= (1 - s - t)\mathbf{p} + s\mathbf{q} + t\mathbf{r} \end{aligned}$$

Strength: create points in a plane

Planes

Family of planes through a point and vector

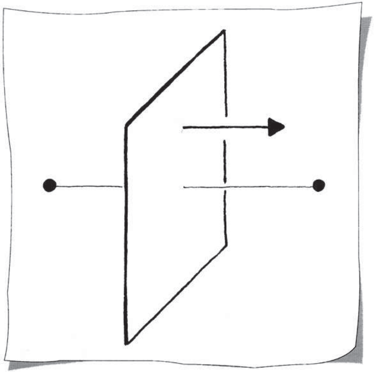


Cannot define plane with one point and a vector in the plane (analogous to implicit form of a plane)

Not enough information to uniquely define a plane
— Many planes fit that data

Planes

A plane defined as the *bisector* of two points



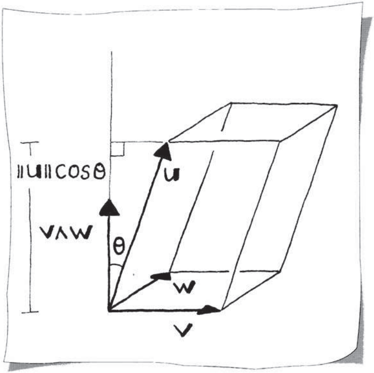
Euclidean geometry definition:
locus of points equidistant from two points

Line between two given points defines the normal to the plane
The midpoint of this line segment defines a point in the plane

With this information — implicit form most natural

Scalar Triple Product

Volume of a parallelepiped



Area P of a parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$

$$P = \|\mathbf{v} \wedge \mathbf{w}\|$$

Volume is a product of a face area height $\|\mathbf{u}\| \cos\theta$

$$V = \|\mathbf{u}\| \|\mathbf{v} \wedge \mathbf{w}\| \cos\theta$$

Substitute a dot product for $\cos\theta$

$$V = \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w})$$

This is called the scalar triple product — signed volume

Signed volume $V = \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w})$
Sign and orientation of the three vectors:
Let $\mathcal{P}$ be the plane formed by $\mathbf{v}$ and $\mathbf{w}$

- $\cos\theta > 0$: positive volume — $\mathbf{u}$ is on the same side of $\mathcal{P}$ as $\mathbf{v} \wedge \mathbf{w}$
- $\cos\theta < 0$: negative volume — $\mathbf{u}$ is on the opposite side of $\mathcal{P}$ as $\mathbf{v} \wedge \mathbf{w}$
- $\cos\theta = 0$: zero volume — $\mathbf{u}$ lies in $\mathcal{P}$—the vectors are *coplanar*

Invariant under *cyclic permutations*

$$V = \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \wedge \mathbf{u})$$

Scalar triple product a fancy name for a 3×3 determinant

— Get to that in Chapter 9

Scalar Triple Product

Example: compute volume for a parallelepiped defined by

$$\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

Compute $\mathbf{y} = \mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ then volume $V = \mathbf{u} \cdot \mathbf{y} = 6$

Notice that if $u_3 = -3$, then $V = -6$

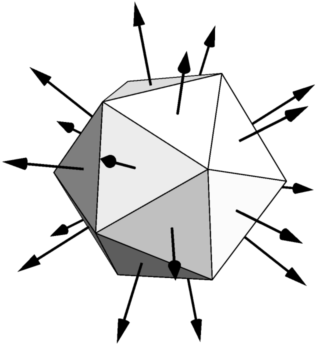
— Sign reveals information about the orientation

Given parallelepiped is simply a $2 \times 1 \times 3$ rectangular box

that has been sheared — Shears preserve volumes: confirms volume 6

Application: Lighting and Shading

Hedgehog plot: the normal of each facet is drawn at the centroid



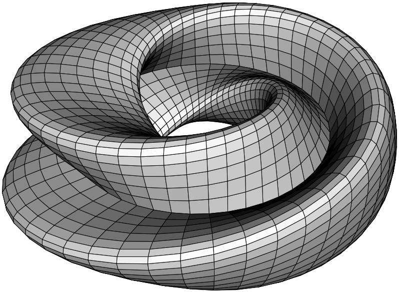
The *normal* to a planar facet: basic element needed to calculate lighting of a 3D object (model)

Normal + light source location + our eye location
$\Rightarrow$ lighting (color) of each vertex

Determining the color of a facet is called *shading*

Application: Lighting and Shading

Flat shading: normal to each planar facets used to calculate the color of each facet



Triangle defined by points $\mathbf{p}, \mathbf{q}, \mathbf{r}$
Form vectors $\mathbf{v}$ and $\mathbf{w}$ from points
Normal

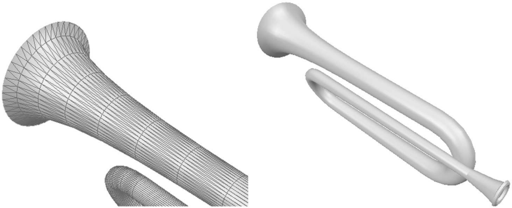
$$\mathbf{n} = \frac{\mathbf{v} \wedge \mathbf{w}}{\|\mathbf{v} \wedge \mathbf{w}\|}$$

By convention: unit length

Consistent orientation of vectors (right-hand rule)
— $\mathbf{v} \wedge \mathbf{w}$ versus $\mathbf{w} \wedge \mathbf{v}$
— Outside versus inside

Application: Lighting and Shading

Smooth shading: a normal at each vertex is used to calculate the illumination over each facet
Left: zoomed-in and displayed with triangles; Right: smooth shaded bugle



At each vertex lighting is calculated: Lighting vectors $\mathbf{i_p}, \mathbf{i_q}, \mathbf{i_r}$
— Each vector indicating red, green, and blue components of light
At point $\mathbf{x} = u\mathbf{p} + v\mathbf{q} + w\mathbf{r}$ assign color

$$\mathbf{i_x} = u\mathbf{i_p} + v\mathbf{i_q} + w\mathbf{i_r}$$

Application of barycentric coordinates

Application: Lighting and Shading

Normals for smooth shading: *vertex normals*
— Simple method: average of the triangle normals around the vertex

Direction of the normal $\mathbf{n}$ relative to our eye's position can be used to eliminate facets from the rendering pipeline
Process called *culling* $\Rightarrow$ Great savings in rendering time
$\mathbf{c}$: centroid of triangle $\mathbf{e}$: eye's position

$$\mathbf{v} = (\mathbf{e} - \mathbf{c})/\|\mathbf{e} - \mathbf{c}\|$$

If $\mathbf{n} \cdot \mathbf{v} < 0$ then triangle is back-facing

WYSK

- 3D vector
- 3D point
- vector length
- unit vector
- dot product
- cross product
- right-hand rule
- orthonormal
- area
- Lagrange's identity
- 3D line
- implicit form of a plane
- parametric form of a plane
- normal
- point-normal plane equation
- point-plane distance
- plane-origin distance
- barycentric coordinates
- scalar triple product
- volume
- cyclic permutations of vectors
- triangle normal
- back-facing triangle
- lighting model
- flat and Gouraud shading
- vertex normal
- culling