Anamorphic 3D geometry

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Abstract

An anamorphic image appears distorted from all but a few viewpoints. They have been studied by artists and architects since the early fifteenth century. Computer graphics opens the door to anamorphic 3D geometry. We are not bound by physical reality nor a static canvas. Here we describe a simple method for achieving anamorphoses of 3D objects by utilizing a variation of a simple projective map that is well-known in the computer graphics literature. The novelty of this work is the creation of anamorphic 3D digital models, resulting in a tool for artists and architects.

1. Introduction

An *anamorphic image* appears distorted from all but a few viewpoints. The term comes from Greek *ana-*, again and *morphe-*, shape. Figure 1 illustrates an early example of an anamorph (short for anamorphosis) in the painting "The Ambassadors" by Hans Holbein the Younger (1536). At the feet of the ambassadors is an unrecognizable object contrasting the rest of the painting which is composed of clearly and precisely rendered elements. This object is a skull, illustrated in a corrected form in the right part of the figure. It can be seen by moving close to the wall at the right side of the picture plane.[1]

One could argue that anamorphs were first constructed in Roman times as "accelerated" and "decelerated" perspective, whereby structures such as columns were built with non-standard dimensions in order to appear farther or nearer from an observer than they were in reality. Greek and Roman artists used the technique of *tromp-l'oeil*[2] (deceive the eye) to make the viewer believe they are seeing something three-dimensional when in reality the work is only two-dimensional.

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[1] Kent [13] offers a "true" restoration created using a digital camera and a poster reproduction of the painting. Baltrusaitis [2] dedicates a chapter to this painting.
[2] Although the term did not appear until the 17th century.

Fig. 1. "The Ambassadors" by Hans Holbein the Younger was painted in 1536. The figure floating in the foreground of the painting is an anamorphic projection of a skull. The "corrected" image of the skull, shown on the right, can be seen by moving close to the wall at the right side of the picture plane

The curious effects of anamorphs as they are known today, were first understood and explored by Leonardo Da Vinci who included anamorphic drawings of a child's head in his Codex Atlanticus (ca. 1485). The appearance of anamorphs as a consciously applied technique in the history of art is nearly simultaneous with the restoration of the study of perspective in the Renaissance period (early 15th century) by artists and architects such as F. Brunelleschi and L. Alberti [2], [7], [12].

In the seventeenth century, S. de Caus and J.-F. Niceron, most notably, dedicated themselves to the study of perspective which included anamorphs. Figure 2, created by Niceron (1638), illustrates the method for creating an anamorph. This is an application of the *construzione legittima*, a trapezoidal checkerboard, going back to Alberti and da Vinci. The original image is placed in a regular grid. A base edge for a trapezoidal grid is selected with the same partition as the corresponding edge in the regular grid. A finite horizon line is selected. On this line there are two points of interest: one (principal or eye point) at which the horizontal lines in the regular grid converge and one at which one set of diagonal lines converge. This determines the partition of the trapezoidal grid to which the image can be transferred [2].

The study of perspective, its application and place in mathematics, continued through the centuries. For example, in the nineteenth century, La Gournerie [14] studied geometric distortions of architectual plans and Staudigl [19] studied the correspondence of orthographic and perspective projections from special viewpoints.
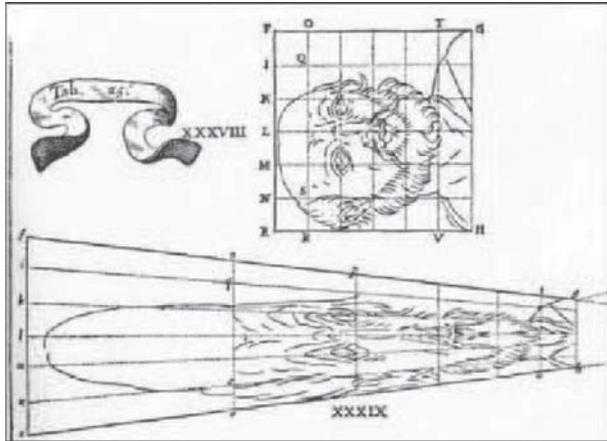
Fig. 2. Demonstration of the method of creating an anamorph by J.-F. Niceron 1638. See [2]

Anamorphs are categorized into two groups. Optical anamorphs can be seen from a specific point of view without any sort of apparatus. (This is the sort we will be concerned with here.) Catoptric or mirrored anamorphs require a mirror of some kind, often cylindrical or conical. A wealth of information on the various anamorphic techniques may be found in Kent [13] and Hunt et al. [11]. Noteworthy is Beever [3], who is known for vivid and captivating anamorphic pavement drawings. We are all familiar with road markings such as bike symbols or arrows that are drawn to compensate for forshortening observed by an approaching driver [13].[3] Additional insights into anamorphs may be found in Andersen [1], Baltrusaitis [2], Clarici [6], Collins [7], Gardner [9], Hunt et al. [11], Kemp [12], Leeman [15], and Walker [21].

Figure 3 illustrates one of Collins' "real" experiments with anamorphs. A mask of George Washington was scanned, and then the 3D digital model was deformed by moving groups of vertices, so as to create an (approximate) anamorph. This tedious process led the authors to look for an easier way to define anamorphs for *3D geometry*.

3D computer graphics opens new doors to anamorphs. We are not bound by physical reality nor a static canvas. Here we describe a simple method for achieving anamorphs of 3D objects by utilizing a simple projective map (collineation), well-known in the computer graphics literature [8], [18], [20], that takes a frustum to an "orthographic box". The method presented here is equivalent to the methods employed by Niceron and his contemporaries. The novelty of this work is the creation of anamorphic 3D digital models, and the realization that a commonly known map can be used to create anamorphs for 3D digital models. Additionally, we present an analytic tool for artists and architects. We plan to pursue more anamorphic projects with Kreysler and Associates (see Fig. 3), a company that specializes

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[3] These markings are linearly stretched, and as we see from Fig. 2, an anamorph is a rational linear operation.
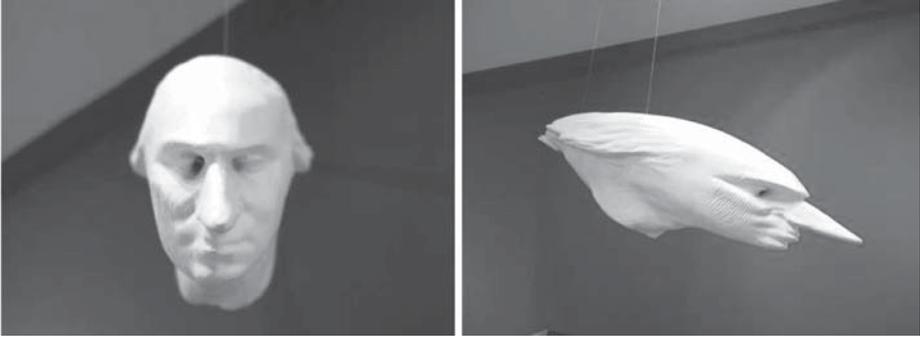
Fig. 3. Anamorph of George Washington. A mask was scanned, and then the 3D digital model was deformed so as to create an (approximate) anamorphic version, seen on the left from an undistorted viewpoint. On the right is a side-view of the anamorphic mask. It was created in foam by Kreysler and Associates, http://www.kreysler.com/projects/George Washington/georgewashington.htm

in architectural facades. In addition, this method is a contribution to the many volume deformation tools in computer graphics and geometric modeling literature [4], [10], [17].

The development of anamorphic 3D geometry proceeds as follows. Section 2 introduces elements of the graphics pipeline. We contend that an understanding of this pipeline facilitates an understanding of the classical method for creating anamorphs. We observe that a common, well-known element of the pipeline can be used to create interesting, anamophic deformations of 3D objects. Section 3 derives this element, a projective map known as a collineation. Section 4 develops and outlines the technique. A demonstration of the method is provided in Sect. 5. In Sect. 6, we present conclusions and ideas for future work.

2. The graphics pipeline and projective maps

The graphics pipeline consists of several coordinate transformations designed for optimization of the algorithms applied to a primitive traveling down the pipeline [20]. The first such transformation is to place the geometry in *eye coordinates*: consider your eye positioned at the origin, looking at your object positioned on the $-z$-axis. The next step is to define a *viewing volume* around your geometry which will determine what is displayed. Geometry outside of the viewing volume will be clipped (eliminated). The projection method that we choose plays a role in the specification of this viewing volume.

Orthographic and perspective projections, illustrated in Fig. 4, are the methods most often used in computer graphics. At the end of the pipeline, the object will be projected into a *view plane* in front of your eye. An orthographic projection projects all vertices in a direction perpendicular to the view plane. A perspective projection
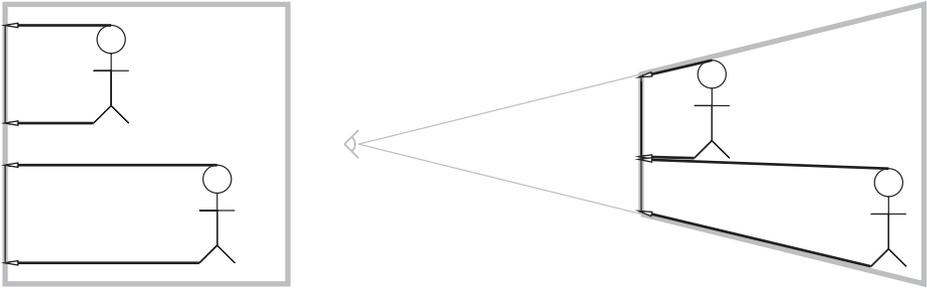
Fig. 4. Left: an orthographic projection. Vertices are projected perpendicular to the view plane. Right: a perspective projection. Vertices are projected into the view plane through the center of projection, resulting in foreshortening
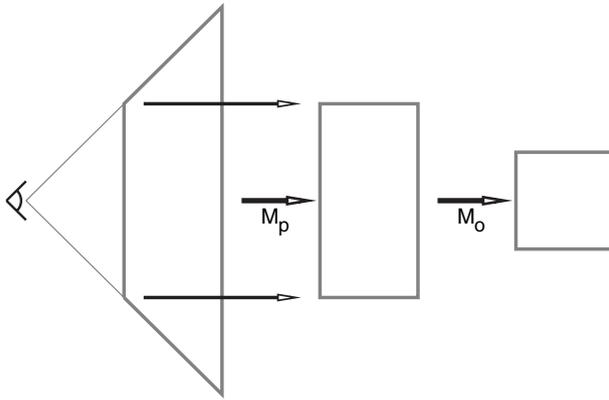
Fig. 5. With a perspective projection, the transformation from eye to clip coordinates involves two mappings. First M_p maps the frustum to a box sharing the dimensions of the view plane and z-depth of the frustum. Next this box is mapped to a cube with edge length two, centered at the origin

projects a vertex along a line defined by that vertex and the eye, or *center of projection (cop)*. Thus the viewing volume takes the form of a box for an orthographic projection and a *frustum* (truncated pyramid) for a perspective projection.

Clipping against a frustum is more difficult than clipping against a box. Therefore, a projective map M_p is constructed to map the points in the frustum to a box [5]. This map is special in that "relative" z-depths are preserved,[4] but changed so that an orthographic projection will produce a perspective image. Not only does this facilitate clipping, but also the z-buffer algorithm for hidden surface removal. The viewing volume (now a box) is mapped via an affine map M_o to a normalized box centered at the origin for the purpose of simplify the clipping algorithm. Figure 5 illustrates.

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[4] A more precise definition of this statement follows in (2).
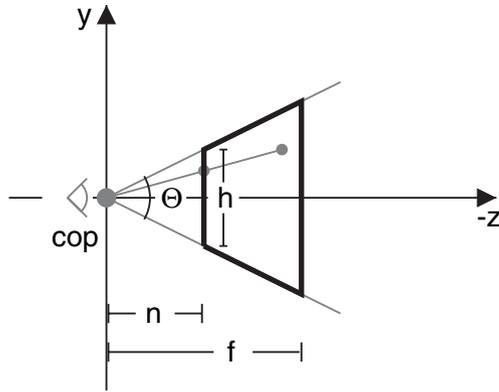
Fig. 6. The perspective projection frustum is defined by the parameters θ, n, f, h and the width of the view (near) plane

3. The projective map: frustum to box

The frustum is illustrated in Fig. 6. The center of projection is at the origin. The angle θ is called the *field of view*. The distance from the eye to the near (or view) plane is n and the distance from the eye to the far plane is f.[5] The height of the near plane is denoted by h. The parameters, θ, h, n, are dependent, and they are related by

$$\tan(\theta/2) = \frac{h/2}{n}. \tag{1}$$

The width w of the view plane completes the frustum's definition.

Homogeneous coordinates serve the purpose of expressing affine and projective maps in a convenient matrix representation [16]. An affine point and corresponding homogeneous point have coordinates

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{p}} = \begin{bmatrix} p_x \cdot w \\ p_y \cdot w \\ p_z \cdot w \\ w \end{bmatrix},$$

respectively.

Projective maps preserve the cross ratio, cr, of four collinear points, and this fact may be used to derive the map M_p. We will use the following definition. Suppose $a, b, c, d \in I\!\!E^1$, then

$$\mathrm{cr}(a, b, c, d) = \frac{\mathrm{ratio}(a, b, d)}{\mathrm{ratio}(a, c, d)}, \quad \text{where} \quad \mathrm{ratio}(a, b, d) = \frac{\mathrm{vol}(a, b)}{\mathrm{vol}(b, d)},$$

and "vol" refers to one-dimensional signed volume [8].

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[5] Since the eye is placed at the origin, n and f will be referred to as both $-z-$values and distances.

To map a point $\mathbf{p} \in I\!\!E^3$ in the frustum to a point $\mathbf{b} \in I\!\!E^3$ in the corresponding orthographic box, we first examine the action of the map in the z-coordinate. The projective map sends the center of projection to infinity, and points in the near and far planes should remain there, respectively. Thus, p_z in the frustum is mapped to b_z in the box by satisfying the following equality:

$$\mathrm{cr}(\mathrm{cop}, n, p_z, f) = \mathrm{cr}(-\infty, n, b_z, f). \tag{2}$$

As illustrated in Fig. 7, determining the x- and y-coordinates is simply a matter of observing that the point must live on the line perpendicular to the near plane, through the point of projection. Employing the rule of similar triangles, the corresponding affine point is

$$\mathbf{b} = \begin{bmatrix} p_x n / p_z \\ p_y n / p_z \\ n + f - f n / p_z \end{bmatrix}. \tag{3}$$

The left most graph in Fig. 8 illustrates the action of this map in the z-coordinate. Points on the near and far planes remain in their respective planes, although the x- and y-values are scaled in the far plane. In fact, the scaling in x and y is a function of z (the distance from the eye). As illustrated in Fig. 5, points in the frustum will be mapped to points in the box sharing the same dimensions as the frustum's near plane and z-extents.
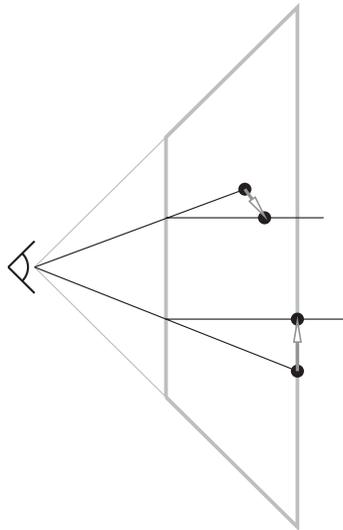
Fig. 7. Action of the projective map: The line through a point's position after perspective division and perpendicular to the near plane determines its new x-and y-coordinates. The z-coordinate is determined by (3). Except for at the near and far plane, points move away from the eye

In matrix form, the projective map M_p takes the form [20], [18]:

$$M_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (n+f)/n & -f \\ 0 & 0 & 1/n & 0 \end{bmatrix}.$$

This map transforms collinear points to collinear points, and it transforms coplanar points to coplanar points, and thus it is also known as a collineation [8]. For this particular construction, the inverse exists, namely

$$M_p^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & n \\ 0 & 0 & \frac{-1}{f} & \frac{f+n}{f} \end{bmatrix}, \tag{4}$$

and it too is a collineation.

4. Anamorphs

In the spirit of Fig. 2 (Niceron), creating an anamorph amounts to applying M_p^{-1} given in (4). Now we want to take points, $\mathbf{b}$, in a box to points, $\mathbf{p}$, in a frustum.

Our goal is to create the most dramatic "distortions". One option is to choose the field of view to be large. Also, we allow the near and far planes of the frustum to move, which requires a slight reformulation of M_p. The cross ratio condition in (2) becomes

$$cr(cop, n_p, p_z, f_p) = cr(-\infty, n, b_z, f), \tag{5}$$

where the frustum's near and far plane distances are n_p and f_p, respectively.

The new expression for $\mathbf{b}$ is

$$\mathbf{b} = \begin{bmatrix} p_x n/p_z \\ p_y n/p_z \\ rn_p + f - rn_p f_p/p_z \end{bmatrix}, \quad \text{where} \quad r = \frac{f-n}{f_p - n_p}. \tag{6}$$

The action of this map is similar to that of (3) in that the near and far planes of the frustum are mapped to the near and far planes of the box.

By simply solving for $\mathbf{p}$ in (6), we have the anamorph operation. First establish the new z-value

$$p_z = \frac{rn_p f_p}{rn_p + f - b_z}, \tag{7}$$

and then set the x- and y-values in the frustum:

$$p_x = \frac{b_x p_z}{n} \quad \text{and} \quad p_y = \frac{b_y p_z}{n}. \tag{8}$$

Figure 8 illustrates the z-coordinate action of the map.
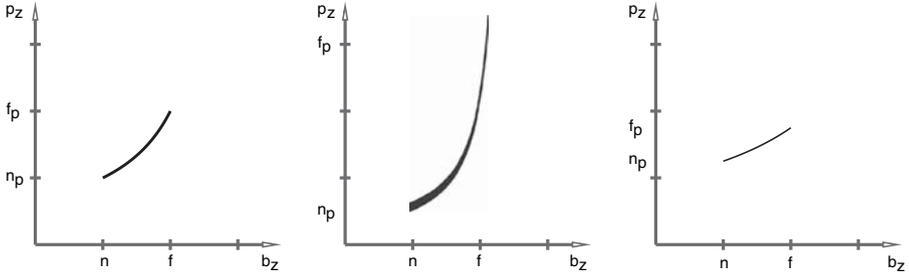
Fig. 8. The rational linear function from (7) illustrates the correspondence between the z-coordinates of points in a box to points in a frustum. Left: $n_p = n$ and $f_p = f$. Middle: $n_p < n$ and $f_p > f$. Right: $n_p > n$ and $f_p < f$

A procedure for creating anamorphic 3D geometry is as follows. Given is a 3D object defined by vertices $\mathbf{b}_i$.

(1) Define a box, such as a bounding box, around the vertices $\mathbf{b}_i$.
(2) Choose one face of this box. The center of projection will be placed in front of this face, and this face will serve as the near plane. Let the distance from this near plane to the opposing face be d.
(3) Specify a field-of-view angle, θ, thereby determining the near distance n from (1).
(4) Assign the far distance of the box: $f = n + d$.
(5) Input a near and far distance n_p and f_p, respectively, for the frustum. (It is convenient to input these relative to n and f so it is not necessary to generate actual values.)
(6) Transform the vertices to eye coordinates.
(7) Apply (7) and then (8) to each vertex of the model.
(8) (Optional) Return the vertices to their original coordinate frame.

This procedure eliminates concern for singularities as long $0 < \theta < 180°$.

5. Demonstration

First we demonstrate our tool on the George Washington (GW) data set, which was the motivation for this work in the first place. In the top row of Fig. 9 is the original data set, and in the bottom row is the anamorphic data set. On the left, we choose a viewpoint (close to the projection center) where GW looks the same in both data sets. The original data set is displayed with an orthographic projection and the anamorphic data set is displayed with a perspective projection.[6] In the right-bottom

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[6] This idea is not new; La Gournerie [14], and Staudigl [19] studied the correspondence between orthographic and perspective images from certain viewpoints.
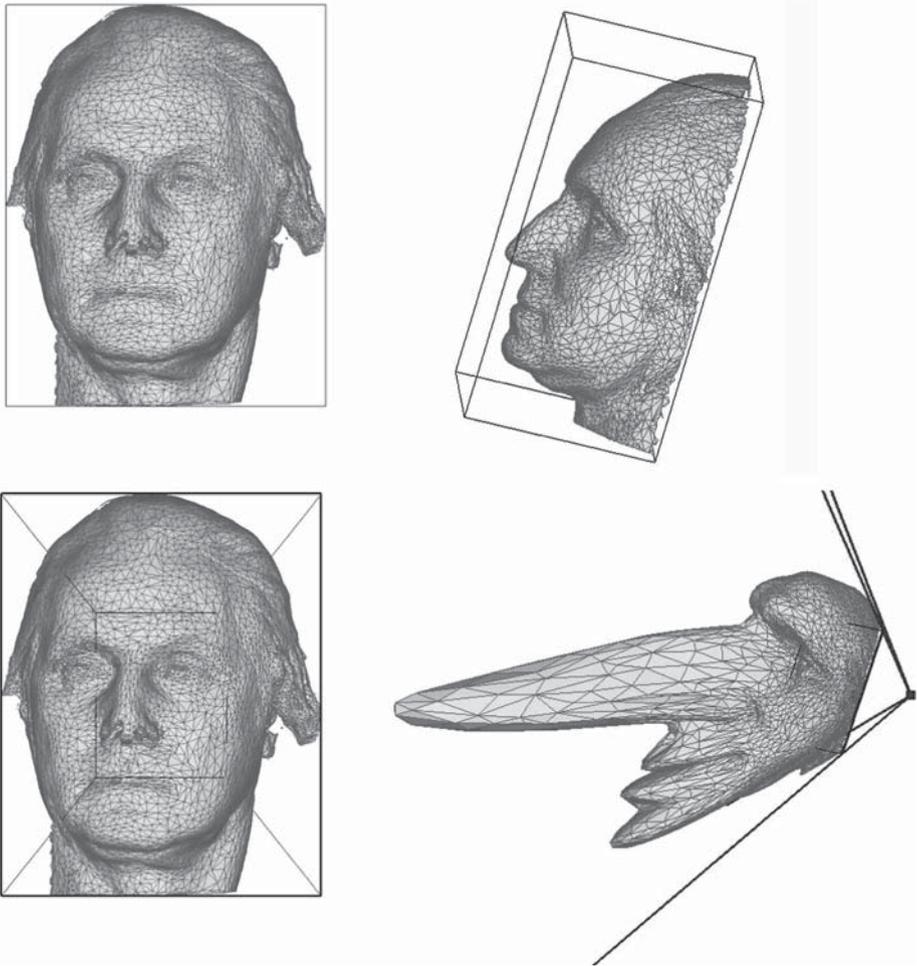
Fig. 9. Anamorphic George Washington: top row illustrates the original data set and the bottom row illustrates the anamorphic data set with $\theta = 100°$, $n_p = n$, and $f_p = 4f$

figure, we can see the anamorph with a field of view $\theta = 100°$, $n_p = n$, and $f_p = 4f$. In the right part of the figure, both images are created with an orthographic projection. In the left part of the figure, we are actually seeing GW from the backside, as is made clear by the location of the frustum in the bottom-right figure.[7]

To make the bottom-left figure appear identical to the top-left figure as *shaded* images, one must use the original data set's normals for the anamorphic data set. And we have done this to further the illusion! Of course the wireframe models look identical, and so we have included them in the rendering. The bottom-right figure is rendered with its true normals.

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[7] Due to zooming-in on the geometry, parts of the frustum have been clipped.

The layout of Fig. 10 follows the same format as Fig. 9. The anamorph has been constructed with $\theta = 80°$, $n_p = n/2$, and $f_p = 3f$. Because the frustum's near plane is closer to the eye than the box's near plane, the teapot is pushed toward the eye, and the handle distortion is accentuated. This same behavior is illustrated in Fig. 11 where the eye has been placed on the opposite side of the teapot, $\theta = 80°$, $n_p = n/4$, and $f_p = f$. The middle plot in Fig. 8 shows that the configuration of near planes used in these teapot examples causes a slow movement of z (away from the eye) close to the frustum's near plane.

A few comments on the anamorph procedure's parameters are in order. Making the field of view small (e.g., $\theta = 1°$) in the procedure in Sect. 4 results in a large near distance n. In the resulting anamorph with $n_p = n$ and $f_p = f$, there is nearly no change in the data since the frustum has nearly the same shape as the box. On the other hand, large field of views result in more distortion. It should be clear that the orientation of the box has a significant impact on the resulting anamorph. In Fig. 9, we see that the tool would be improved with the addition of adaptive meshing for smoothing the deformed model.
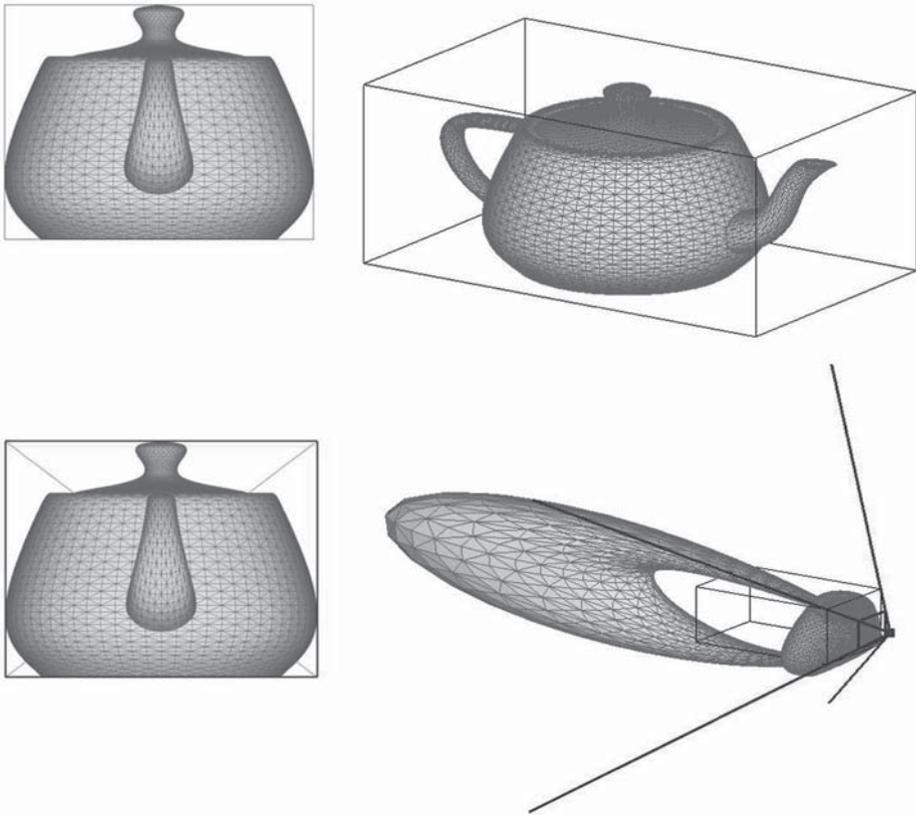
Fig. 10. Anamorphic teapot: top row illustrates the original data set and the bottom row illustrates the anamorphic data set with $\theta = 80°$, $n_p = n/2$, and $f_p = 3f$
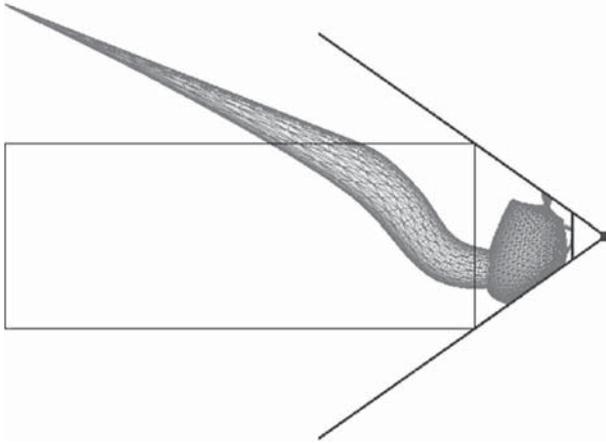
Fig. 11. Anamorphic teapot: A spout for reaching across the table! The parameters for the anamorphic data set include $\theta = 80°$, $n_p = n/4$, and $f_p = f$. The rectangle is the orthographic box

6. Conclusions

We have presented a tool for creating anamorphic 3D geometry. We have remarked that this tool, a collineation, was already familiar to those in computer graphics and mathematics. The novelty here is that we have applied this to 3D triangle meshes. We plan to use this tool for building anamorphic sculptures similar to that in Fig. 3. Additionally, this tool is a contribution to the many volume deformation tools in computer graphics literature [4], [10], [17].

Future work on anamorphs of 3D models will include investigating what can be done with conical and cylindrical anamorphs in combination with their respective reflecting surface and a ray tracing rendering. Anamorphic texture mapping might be interesting to persue also. Other collineations could be explored.

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