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Curves with quadric boundary precision

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Abstract

We describe a method for constructing rational quadratic patch boundary curves for scattered data in $\mathbb{R}^3$. The method has quadric boundary precision; if the given point and normal data are extracted from a quadric, then the boundary curves will lie on this quadric. Each boundary curve is a conic section represented in the rational Bézier representation.

Keywords: Quadrics; Bézier triangles; Scattered data interpolation; rational curves

1. Introduction

Generating boundary curves is often the first step in constructing a scattered data interpolant with respect to triangulated data in $\mathbb{R}^3$. Here a boundary curve scheme is described which has *quadric boundary precision*; if the given data comes from one quadric Q, then the boundary curves will lie on Q.

In the construction of the boundary curves, it is assumed that the following information is available.

- Data points $p_i \in \mathbb{R}^3$.
- A normal $n_i \in \mathbb{R}^3$ at each p_i.
- A triangulation[1] $\mathcal{T}$ of the p_i (cf. Choi et al. (1988)).

Each boundary curve interpolates to the point and normal information. A conic is constructed between each pair of p_i and p_j which are joined by an edge in the triangulation. In (Hansford, 1991), this boundary curve scheme is incorporated into a G^1 scattered data interpolant.

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[1] The discussion will be limited to triangulated data, however it is straightforward to modify the algorithm to data in a quadrilateral structure.

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One of the basic building blocks of quadrics are conics; any planar intersection of a quadric is a conic. Therefore, conics are used as the building blocks for this boundary curve scheme which has quadric boundary precision. Each boundary curve is a conic represented as a rational quadratic Bézier curve.

The first section presents the basic tools and notation used in the boundary curve scheme. The presentation of the boundary curve scheme in the second section proceeds in the following manner. First, the scheme is developed for data which comes from one irreducible quadric. Second, the scheme is developed for data which is not from a quadric. Possible degeneracies are described in the succeeding subsection. In the last section, there is a discussion of the boundary curves used in the context of a rational G^1 interpolant. Additionally, remarks and open problems are presented.

2. Preliminaries

2.1. *Rational quadratic Bézier curves on conics*

A quadratic rational Bézier curve $\boldsymbol{x}(t)$ is given by

$$\boldsymbol{x}(t) = \frac{\sum_{i=0}^{2} w_i \boldsymbol{b}_i B_i^2(t)}{\sum_{i=0}^{2} w_i B_i^2(t)}, \quad \boldsymbol{x}(t), \boldsymbol{b}_i \in \mathbb{E}^3. \tag{1}$$

The $\boldsymbol{b}_i$ are the *Bézier control points* and the w_i are referred to as *weights*. The B_i^2 are the familiar quadratic Bernstein basis functions.

Every rational quadratic Bézier curve defines a conic, cf. (Farin, 1992; Faux and Pratt, 1979; Lee, 1987). A conic is determined by five coplanar pieces of information. The shape (or graph) of a rational quadratic Bézier curve is also determined by five pieces of information: two points with tangents and a ratio $k = 4w_1^2/w_0w_2$ (cf. (Lee, 1987)).

By considering $\rho^{2-i}w_i$, $\rho \in \mathbb{R} - \{0\}$ as the weights of $\boldsymbol{x}(t)$ in (1), we have a way to change the weights but not change the graph of the curve. This process of changing the parameter value associated with each point on the curve is called *reparametrization.* One of the most common utilizations of reparametrization arises when the end weights (w_0 and w_2) are set to unity. If this is performed, the curve is in *standard form.*[2] In the context of a standardized curve (with positive weights), the choice of $\rho = -1$ is of special interest. This reparametrization will allow the *complementary segment* to be traced as the parameter $t \in [0, 1]$. The complementary segment is the portion of the conic which does not lie within the convex hull of the control polygon.

2.1.1. *A problem*

A commonly occurring problem is the following. *Given two points and corresponding tangents and another point, find the interpolating conic.* It is important to remember that the information *given* in this problem is sufficient to determine the shape of the

[2] It is not always possible to represent a conic in standard form.

conic, it is not enough information to determine the parametrization of the rational formulation of the conic. Following the work of Farin (1992) and Lee (1987), we choose $t = 1/2$ as the parameter value to be assigned to the given (additional) point. The Bézier polygon is formed by the given point and tangent information. With this assumption, the relationship between the barycentric coordinates τ_i of the given point with respect to the polygon and weights w_i is

$$w_0 : w_1 : w_2 = \tau_0 : \tfrac{1}{2}\tau_1 : \tau_2. \tag{2}$$

If the curve is standardized, our given point will no longer be associated with the parameter value $t = 1/2$.

If $0 < \tau_0, \tau_1, \tau_2 < 1$ then determining w_1 in (2) with standardization is not a problem. There are special sets of τ_i that must be handled differently. There are three cases. When $\tau_1 = 1$, the conic consists of two intersecting lines. When $\tau_0 = 0$ or $\tau_2 = 0$, the conic consists of two parallel lines. When $\tau_0 = 1$ or $\tau_2 = 1$, the conic consists of two parallel lines.

2.2. *Quadric surfaces*

Quadrics are surfaces that are quadratic in three variables,

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0. \tag{3}$$

Apparent from (3), a quadric is determined by nine scalar-valued coefficients. This means that one can specify nine points in $\mathbb{E}^3$ that are in general position, and an *irreducible quadric* can be found that will pass through these points. For the points to be in general position: no seven are coplanar, no six lie on one conic, no four are collinear, and no two are coincident. There are nine irreducible quadrics. We will disregard all other quadrics.

Any tangent plane of a quadric meets the quadric in a pair of lines, all together forming two families of so-called *generatrices*.

- If both families are real and different, the quadric is said to be *doubly ruled* or *annular* and is a hyperboloid of one sheet or a hyperbolic paraboloid.
- If both families are non-real but different, the quadric is said to be non-ruled or *oval* and is an ellipsoid, a hyperboloid of two sheets, or an elliptic paraboloid.
- In the special case, where both families coincide to one family, the quadric is said to be *singly ruled* or *degenerate* and is a quadratic cone or an elliptic, parabolic or hyperbolic cylinder. Note that any tangent plane of a singly ruled quadric touches the surface along a whole generatrix.

3. Boundary curve method

3.1. *Boundary curves for data from a quadric*

This boundary curve scheme was developed with the intent to have quadric boundary precision. Therefore, we choose to first present the scheme for data from a quadric, so that the rationale of each step may be developed. However, the scheme does not use knowledge of the surface from which the data was taken.

The boundary curves generated by this scheme are conics represented as rational quadratic Bézier curves. There are three fundamental steps to this boundary curve scheme. First, the plane in which a boundary curve lies is determined. This plane is referred to as the *boundary plane*. Second, the middle Bézier point, $\boldsymbol{b}_1$ of the rational quadratic polygon is constructed. Third, weights are assigned to each Bézier point.

The next subsections describe these steps. Only one edge in the triangulation is discussed, as all edges are treated similarly.

3.1.1. The boundary plane

Two points, $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ with normals, $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$, (or equivalently, tangent planes) are the given information. There are an infinite number of conics which lie on Q and pass through $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$. Each of these conics lies in a plane that passes through $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$. Therefore, by choosing one of these planes, the shape of the boundary curve is determined. It is important that the plane chosen is does not depend upon the ordering or indexing of the data points. In other words, the same boundary plane with respect to an edge in the triangulation should be produced irrespective of which triangle is currently under consideration.

Two planes are apparent. The plane formed by $\boldsymbol{p}_0$, $\boldsymbol{n}_0$ and $\boldsymbol{p}_1$, and the plane formed by $\boldsymbol{p}_1$, $\boldsymbol{n}_1$ and $\boldsymbol{p}_0$. Thus the method proposed for choosing a boundary plane is to "average" these planes. This is done by first constructing a point

$$\hat{\boldsymbol{p}} = \tfrac{1}{2}(\boldsymbol{p}_0 + \boldsymbol{p}_1) + (\boldsymbol{n}_0 + \boldsymbol{n}_1). \tag{4}$$

Then, the three points $\boldsymbol{p}_0$, $\hat{\boldsymbol{p}}$, and $\boldsymbol{p}_1$ define the desired boundary plane. The same scheme was used by Piper (1987) and Hamann et al. (1991). If a different choice of boundary plane is used then quadric boundary precision will still be obtained.[3] Clearly, degenerate cases can occur in this definition of the boundary plane. These are discussed in Section 3.3.

3.1.2. The rational quadratic polygon and weights

As illustrated in Fig. 1, at this point we have two points ($\boldsymbol{p}_0$ and $\boldsymbol{p}_1$) with tangent planes ($\boldsymbol{T}_0$ and $\boldsymbol{T}_1$), and a boundary plane $\boldsymbol{B}$. The intersection of $\boldsymbol{B}$ and Q is a conic. The first step in representing this conic in rational quadratic Bézier form, $\boldsymbol{c}(t)$, is to determine the *polygon*: $\boldsymbol{b}_0$, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. The endpoints pose no problem; simply set $\boldsymbol{b}_0 = \boldsymbol{p}_0$ and $\boldsymbol{b}_2 = \boldsymbol{p}_1$. (The order here is unimportant since the Bernstein basis functions are symmetric.) The middle Bézier point, $\boldsymbol{b}_1$, is simply the intersection of the three planes $\boldsymbol{T}_0$, $\boldsymbol{T}_1$ and $\boldsymbol{B}$.

Given the polygon, the next step is to determine the *weights* which define the conic, $\boldsymbol{c}(t)$, which lies on Q in $\boldsymbol{B}$. As discussed in Section 2.1.1, it is a straight forward process to determine the weights if we are given another point, $\boldsymbol{s}$, on the conic.

[3] The above method localizes the determination of the boundary plane. Other, less local, schemes are conceivable, although they are likely to be more expensive.

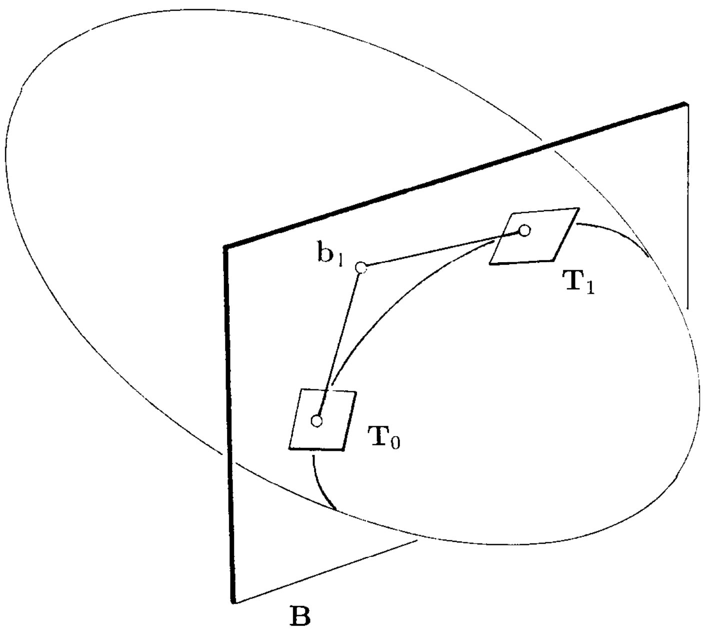

Fig. 1. Two points and tangent planes (T_0 and T_1) on a quadric, and the boundary plane B. The boundary plane intersects the quadric in a conic section, which as shown can be represented as a rational quadratic Bézier curve.

The key to finding a point s on c is the *slicing conic*. To determine a slicing conic $\hat{c}(t)$ on Q, we first choose three (given) data points, labeled $p_{i_0}, p_{i_1}, p_{i_2}$. At two of these points, say p_{i_0} and p_{i_1}, the tangent planes (T_{i_0} and T_{i_1}) are needed. The term slicing conic comes from the fact that we "slice" Q with the plane S formed by the three points, creating a conic in this "slicing plane. Following Fig. 2, the construction of such a slicing conic includes the following steps.

1. Find the point of intersection of T_{i_0}, T_{i_1} and S; This point is $\hat{b}_1$ of the rational quadratic representation $\hat{c}(t)$ of the slicing conic. Additionally, let $\hat{b}_0 = p_{i_0}$ and $\hat{b}_2 = p_{i_1}$.
2. Since p_{i_2} is on the slicing conic, it can be used to determine the weights of $\hat{c}(t)$. (The curve is then standardized.) Notice that p_{i_2} lies on the complementary segment, causing $\hat{w}_1 < 0$. By simply negating the weight we obtain a parametrization of the portion of the curve within the polygon, $\hat{b}_0$, $\hat{b}_1$, and $\hat{b}_2$.

Recall that the slicing conic $\hat{c}(t)$ is defined for the purpose of determining a point $s = c(1/2)$ on the boundary curve. The point s is realized by constructing $\hat{c}$ such that it intersects the boundary plane B.

A plane and a conic either have no intersections, two (coincident or distinct) intersections, or an infinite number of intersections. Our interest focuses on slicing conics with two distinct intersections with B. Because our data comes from a quadric, we know that if $\hat{c}$ intersects B, then $\hat{c}$ also intersects c there. By choosing p_0 or p_1 as p_{i_0}, then in most instances a point s may be found (that is different from p_0 or p_1). If, for instance,

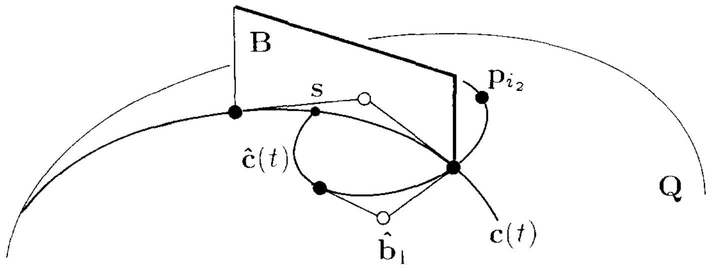

Fig. 2. The construction of a slicing conic and the boundary conic.

the intersections are identically $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ then another slicing conic must be constructed by using a different combination of three points. (Other degenerate cases are discussed in Section 3.3.) By always including $\boldsymbol{p}_0$ or $\boldsymbol{p}_1$, we are assured at least one intersection of $\hat{\boldsymbol{c}}$ with $\boldsymbol{c}$.

With $\boldsymbol{s}$ in hand, we may complete the construction of the boundary curve by assigning appropriate weights (cf. Section 2.1.1). If the point $\boldsymbol{s}$ on $\boldsymbol{c}$ lies outside the convex hull of the polygon, then the weight w_1 must be negated. Thus, the definition of the boundary curve which lies on Q and interpolates the given point and normal information is complete.

3.1.3. Examples

Examples of boundary curves on quadrics are given in Figs. 3–5. One of each quadric type (oval, doubly ruled, singly ruled) is represented. In addition, each conic type appears, including lines as a degenerate conic. The boundary curves are illustrated in the top left of each figure. In some cases, the underlying triangulation is also rendered. In the bottom right of each figure is a Gouraud shaded image of the rational quadratic triangular patches (cf. (de Boor, 1987; Farin, 1986)) formed by the boundary curves.

In each of these figures, the shaded image is hardly distinguishable from the quadric from which the given data was taken. This point is demonstrated further in Fig. 6, with a reflection line analysis of patches on a quadric. However, in general, the patches will be only C^0. Furthermore, in general, a rational quadratic triangular patch does not lie on a quadric. The parametric representation of quadrics with rational quadratic triangular patches, from a geometric point of view, is presented in (Boehm and Hansford, 1991) and (Boehm and Hansford, 1992). An algebraic condition can be found in (Sederberg and Anderson, 1985). In general, rational quartics are necessary, as is shown in (Farin et al., 1988).

3.2. Boundary curves for data from a general surface

If the given data are taken from a general surface type rather than a quadric, then additional precautions must be taken. Importantly, it is not necessary to know a priori the type of surface from which the data has been extracted. Other than the points raised below, the construction is identical to that described in Section 3.1.

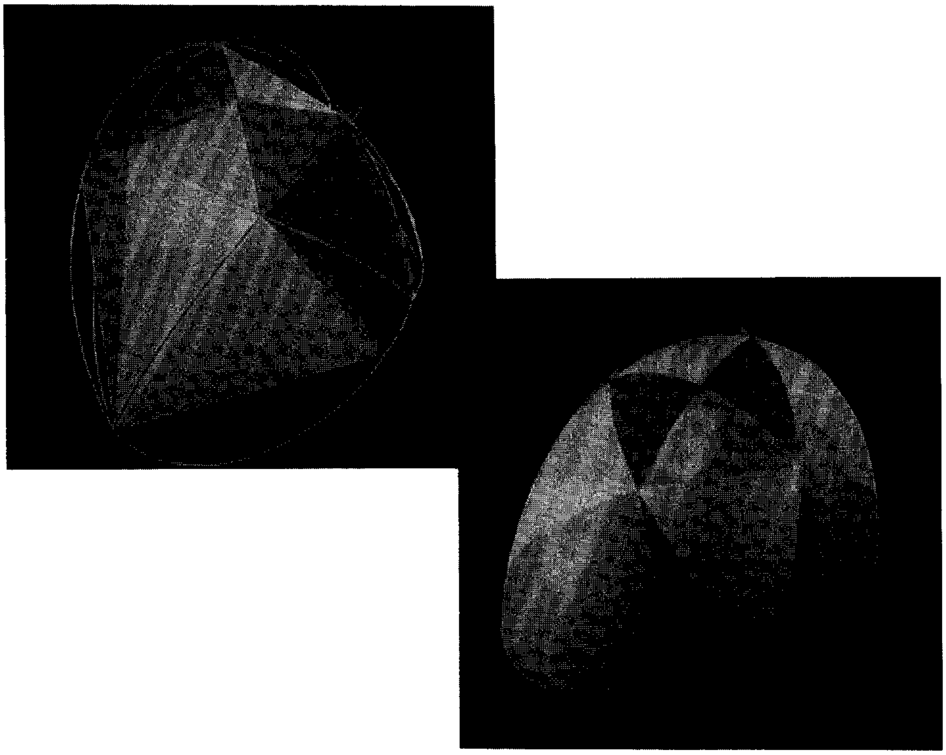

Fig. 3. Boundary curves on an ellipsoid.

The types of degenerate cases that arise is the first way in which the boundary curve scheme behaves differently for data from a general surface S. Particular degeneracies can occur for general data that cannot occur with data from a quadric. Details on the degeneracies are given Section 3.3. It is possible to distinguish between the cases that can and cannot occur for data from a quadric. With this knowledge, each degenerate case can be handled appropriately.

The second way in which the boundary curve scheme behaves differently for data from S is the non-uniqueness of the boundary curve generated over an edge. The choice of slicing conic determines the shape of the boundary curve. For a quadric, the boundary curve is unique once the boundary plane has been constructed, thus all slicing conics yield the same boundary curve. Therefore, if the scheme is executed triangle by triangle for data from S, then an edge could have two boundary curves associated with it. (Of course these conics lie in the same plane and share the same polygon — only the middle weight differs.) Which conic is the best? One choice would be the conic with a weight closest to the weight for the minimum eccentricity conic, cf. (Pratt, 1985; Farin, 1992). In the case of a tie, the smallest weight is preferable: ellipses are generally nicer in shape than hyperbolas.

Illustrated in Fig. 7 is an example of the boundary curve scheme applied to data from a general surface type. Again, the rational quadratic triangular patches formed by the

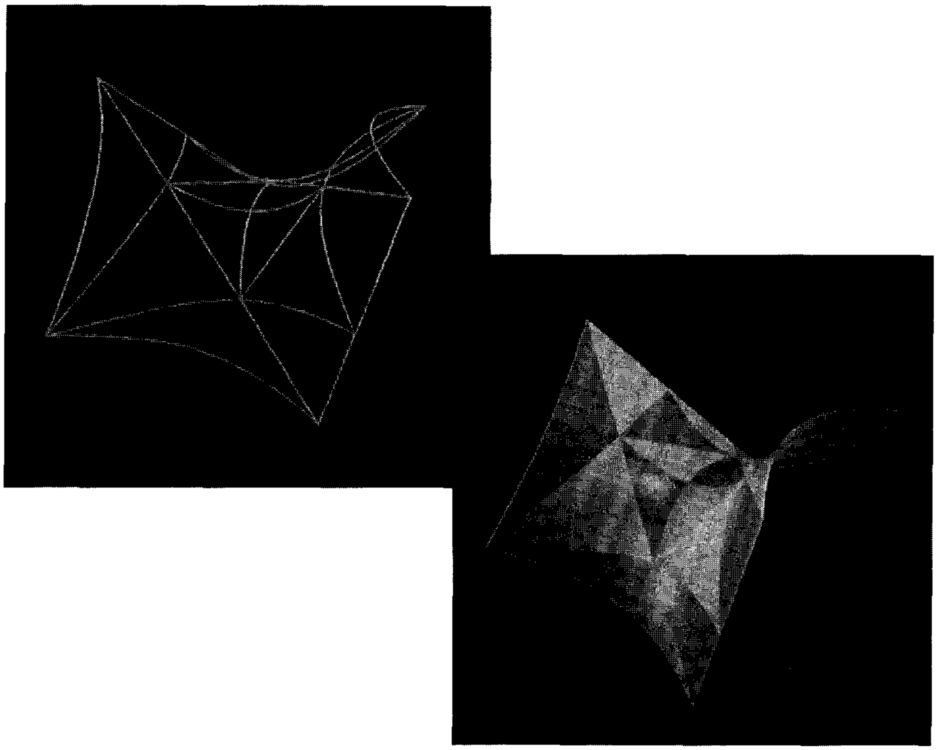

Fig. 4. Boundary curves on a hyperbolic paraboloid.

boundary curves are illustrated.

3.3. Degeneracies

Clearly, an algorithm composed of steps such as intersecting three planes to yield a point, will have "degenerate" cases — instances when special action is needed in order to determine a solution. It is of primary interest to handle all cases such that if the data are from a quadric, then the quadric is reproduced. As presented in the introduction of this section, the three fundamental steps in the scheme include generating a boundary plane, finding a polygon, and then assigning weights to the Bézier points. Degenerate cases occur in each of these steps, as detailed in the subsections below.

3.3.1. Degenerate boundary plane

Degeneracies in the boundary plane generation occur when $\hat{p}$ from (4) becomes collinear with p_0 and p_1. Among these degeneracies, the case when $n_0 = -n_1$ is of special interest. This could happen for arbitrary data, but this could also occur on a quadric, e.g., take the points from the north and south poles of a sphere. We choose to disallow this for boundary curves with quadric boundary precision.

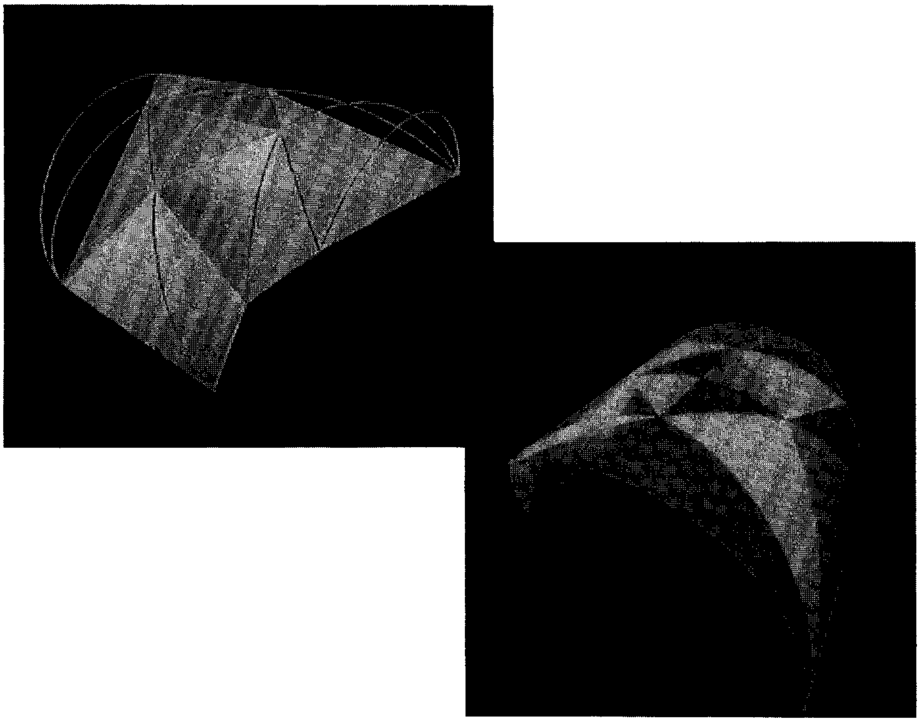

Fig. 5. Boundary curves on a circular cylinder.

3.3.2. Degenerate polygon

The tangents are constructed by intersecting the tangent planes with the boundary plane. It is possible to have tangent information which dictates that a curve with an inflection point is expedient - an impossible task for a rational quadratic. Also posing a problem are tangents that are nearly parallel.

An algorithm has to be developed that decides if a given polygon will generate a curve with or without an inflection point, and the shape of the curve. While the development of such an algorithm is interesting in its own right, it is not discussed here. If an inflection point is called for, an algorithm for constructing two quadratics is described in (Hansford, 1991).

A special (degenerate) case can occur for data from a quadric. Recall that the boundary plane $\boldsymbol{B}$ and two tangent planes, $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$, are intersected to find the middle Bézier point $\boldsymbol{b}_1$. If $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$ intersect in a line which lies in $\boldsymbol{B}$, there are an infinite number of choices for $\boldsymbol{b}_1$. As illustrated in Fig. 8, it is sufficient to place $\boldsymbol{b}_1$ at the midpoint of $\boldsymbol{b}_0$ and $\boldsymbol{b}_2$ and to assign weights of unity to each point — creating a straight line. As in Fig. 8 (top), identical tangent planes imply that the two points lie on the same generator and that the reguli are coincident, hence the quadric is singly ruled. As in Fig. 8 (bottom), either data point lies on a tangent line at the other point: This counts as three collinear points on the quadric, hence the quadric is doubly ruled.

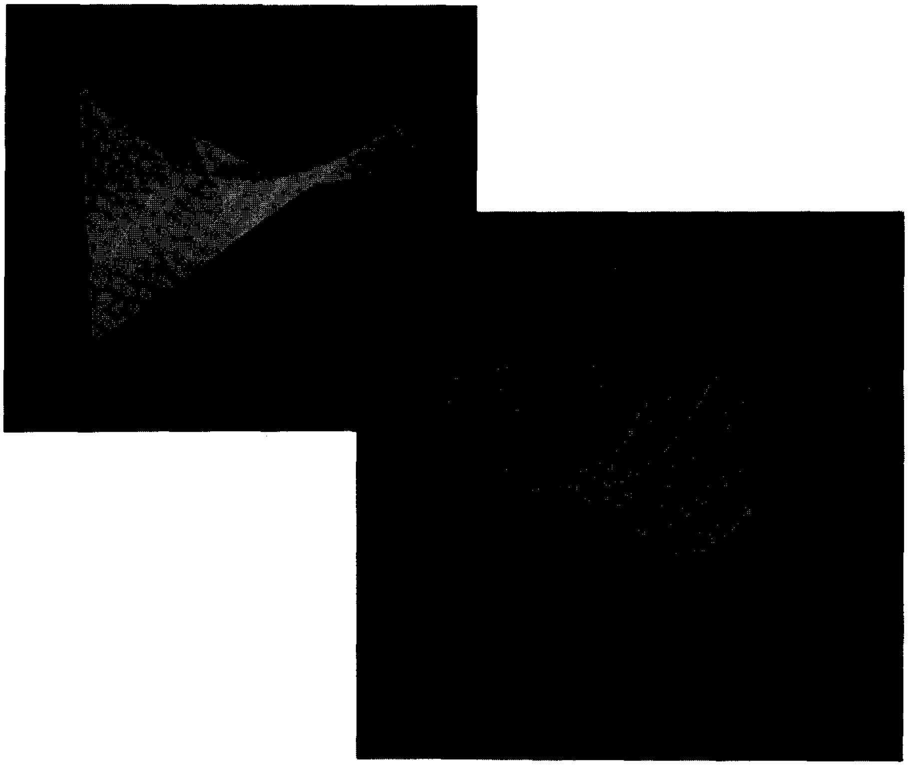

Fig. 6. Reflection lines of quadratic patches on a hyperbolic paraboloid.

3.3.3. Degenerate weights

Given the quadratic polygon and another point on the underlying conic, it was shown in Section 2.1.1 how to determine the weights which yield a rational quadratic representation of the conic. When the given data are not from a quadric, weights arise from this process that are not useful. Before describing how to overcome such a problem, we will show that these cases cannot occur for data from a quadric.

Let $\boldsymbol{b}_0, \boldsymbol{b}_1, \boldsymbol{b}_2$ be the non-collinear control points of the Bézier polygon with associated weights (w_0, w_1, w_2). To find weights, (2) is used; the weights obtained in the following degenerate cases do not allow standardization. The additional point on the conic is referred to as $\boldsymbol{e}$. The first degenerate set of weights is $(w_0, 0, w_2)$. The location of $\boldsymbol{e}$ is collinear with $\boldsymbol{b}_0$ and $\boldsymbol{b}_2$. Suppose the data are from a quadric. For $\boldsymbol{e}$ to be collinear with $\boldsymbol{b}_0$ and $\boldsymbol{b}_2$ means that this line is a generator. A contradiction arises immediately: $\boldsymbol{b}_1$ must be collinear with $\boldsymbol{b}_0$ and $\boldsymbol{b}_2$. Three other similar cases occur when $\boldsymbol{e}$ is either collinear with $\boldsymbol{b}_0\boldsymbol{b}_1$ or $\boldsymbol{b}_1\boldsymbol{b}_2$, or $\boldsymbol{e}$ is identical to $\boldsymbol{b}_1$. None of these cases can occur on a quadric, given the boundary plane generation scheme described in Section 3.1.1; the plane lies halfway between the normals to the surface.

Therefore, if any of the cases of degenerate weights occur, we know that the data are not from a quadric and thus we have some freedom in our solution. We choose to use the minimum eccentricity weight that is implied by the polygon. Before resorting to this

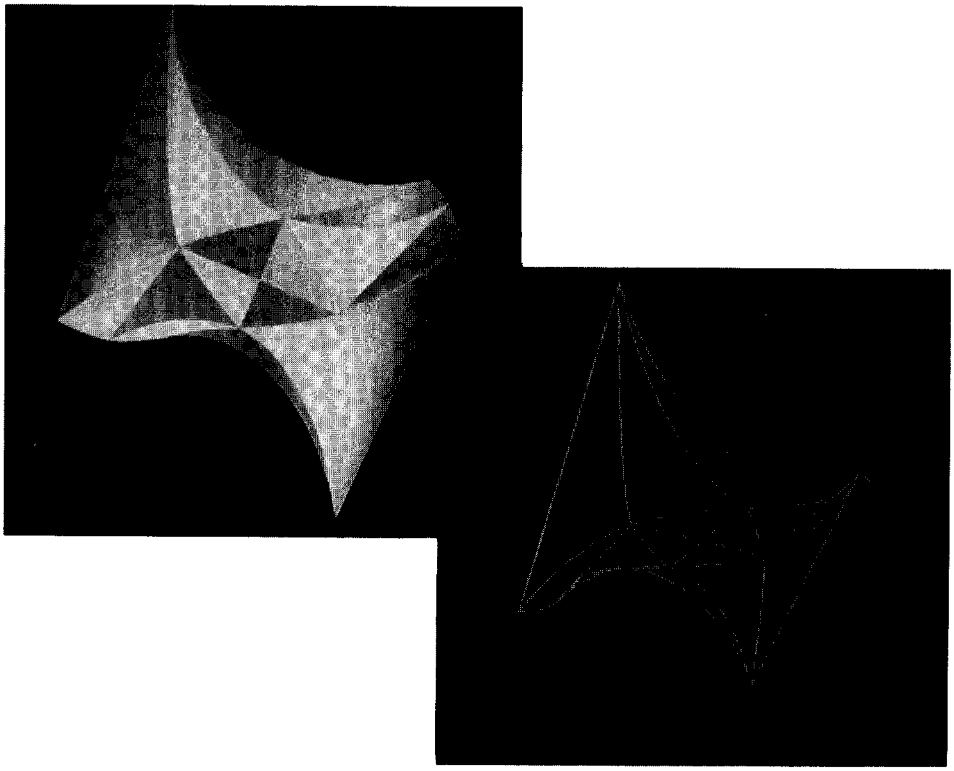

Fig. 7. Boundary curves on a monkey's saddle.

solution, other slicing conics can be constructed that might produce a more satisfactory point. However, after all locally constructed slicing conics are exhausted, the minimum eccentricity weight should be used.

3.3.4. Degenerate slicing conic

In the construction of a slicing conic, degeneracies can occur when the polygon or the weights are generated.

For the polygon degeneracy, it might be that the tangent planes to the surface intersect in a line which is parallel to the slicing plane. Unfortunately, we are unable to deduce the correct location of the middle slicing conic Bézier point, and a new slicing conic must be formed. Notice: if the data are from a general surface S, then by merely changing at which points the tangent planes are given will in general produce a different slicing conic.

For the weight degeneracies, we cannot exclude degenerate cases from happening for data from a quadric, as we did for the boundary conic. This is because the slicing plane is not restricted in its location as is the boundary plane. Degenerate conics, lines, must be constructed. Instead of intersecting a slicing conic with the boundary plane, one or two lines are intersected with the boundary plane.

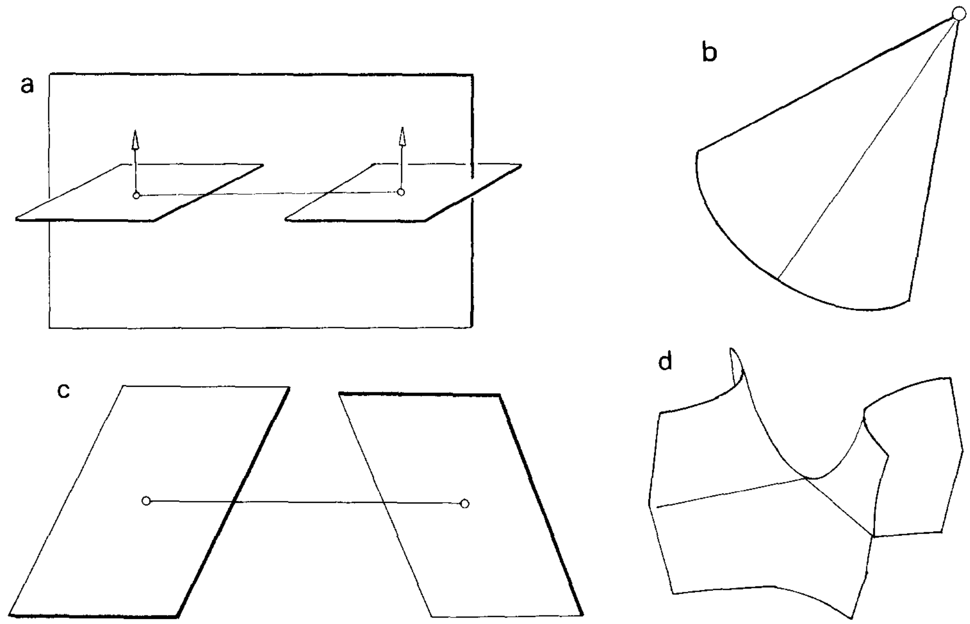

Fig. 8. Left, the intersection of the tangent planes and boundary plane is a line. Right, the type of quadrics on which this can happen. Top: singly ruled. Bottom: doubly ruled.

4. Remarks

In (Hansford, 1991), this boundary curve scheme is incorporated into a G^1 scattered data interpolant. A rational extension of the G^1 polynomial scheme in (Piper, 1987) and (Farin, 1992) is developed. (This scheme uses the Clough-Tocher split.) The patches are constructed to be G^1 in $\mathbb{R}^4$ which ensures G^1 in $\mathbb{R}^3$. This is a more stringent condition than that presented in (Vinacua and Brunet, 1989), which allows the patches to be C^{-1} in $\mathbb{R}^4$. Our rational G^1 scheme occasionally suffered from shape defects, similar to those discussed in (Mann et al., 1992). It remains an open research question to find polynomial or rational G^1 conditions (which are practical to use) that produce surfaces that are free from shape defects.

In principle, the boundary curves presented can be incorporated into any triangular or rectangular scheme. It is not necessary to use them in the so-called split-domain schemes. The boundary curves should simply be degree elevated as necessary. The shape of the boundary curves are quite nice, and seem to satisfy the suggestion of Mann et al. (1992) that the boundary curves should not have flat regions toward the middle. It should be noted that this boundary curve scheme is geared toward data that are monotonic — or quadric like.

Another open question is to determine if a given rational quartic patch actually lies on a quadric. If the quadric is known, this question has been answered in the work by Hoschek (1992) and Dietz et al. (1993).

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