Computer-Aided Design 80 (2016) 6–8

Contents lists available at ScienceDirect

Computer-Aided Design

journal homepage: www.elsevier.com/locate/cad

Curvature combs and curvature plots

Gerald Farin [1]

Computer Science, Arizona State University, P.O Box 878809, Tempe, AZ 85287-8809, USA

A R T I C L E I N F O

Article history:
Received 28 April 2015
Accepted 9 August 2016

Keywords:
Curve design
Curvature
Curvature comb
Frenet frame

A B S T R A C T

We compare curvature combs and curvature plots.

1. Introduction

Many objects in Industrial or Conceptual Design rely on an initial set of "feature curves". These are the essence of the shape information for the object's appearance.

For many applications, these curves are planar, for example flow lines of a ship hull or a grid of cross sections of the roof of a car. For others, they will be 3D space curves, such as the outline of a car's hood.

Either type of curve has to be "fair" or "pleasant", thus giving rise to acceptable surfaces for the object to be designed. The question arises as to what constitutes a fair curve. Most CAD systems seem to rely on some sort of curvature information for this, and the prevailing tools are curvature combs and curvature plots. A comparison of these tools is the topic of the present paper.

2. Definitions

Let $\mathbf{x}(t)$ be a 3D parametric curve. We will use the following quantities from differential geometry (see [1]):

the first two parametric derivatives

$$\dot{\mathbf{x}}(t) \quad \text{and} \quad \ddot{\mathbf{x}}(t),$$

the binormal vector (with $\|\cdot\|$ denoting normalizing to unit length)

$$\mathbf{b}(t) = \frac{\dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t)}{\|\cdot\|},$$

and the normal vector

$$\mathbf{n}(t) = \frac{\mathbf{b}(t) \wedge \dot{\mathbf{x}}(t)}{\|\cdot\|}.$$

3. Curvature plots

The *curvature* of a 3D curve is given by

$$\kappa(t) = \frac{\|\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}\|}{\|\dot{\mathbf{x}}\|^3}.$$

For the special case of planar curves, we may define *signed curvature*

$$\kappa_s(t) = \frac{\det[\dot{\mathbf{x}}, \ddot{\mathbf{x}}]}{\|\dot{\mathbf{x}}\|^3},$$

where det is the determinant. Note that 3D curvature is always nonnegative whereas the 2D case allows for mixed-sign curvatures. Thus for 2D curves, the notion of "inflection point" (sign change of κ_s) makes sense, but it does not for 3D curves.

The graph of $\kappa(t)$ or $\kappa_s(t)$ is referred to as the curvature plot. What makes a "good" curvature plot? It is generally agreed that good plots have only very few monotone segments, corresponding to low frequency behavior of the curvature. High frequency curvature behavior indicates "unpleasant" curves. Also, a good curvature plot lacks "corners" (tangent discontinuities). See [2,3].

4. Combs

The curvature comb of a curve is defined as:

$$\mathbf{c}(t) = \mathbf{x}(t) - d \cdot \kappa(t)\mathbf{n}(t). \tag{1}$$

If we discretize $\mathbf{c}$, then we get the appearance of a comb distributed along $\mathbf{x}$, as illustrated in Fig. 1, hence the term "curvature comb". In our examples, comb curves are shown in red; the input curves are in black. The scaling factor d is needed to bring out the salient features of the comb.

[1] Died 14 January 2016. Final manuscript communicated by George Allen (george.allen@siemens.com) and Dianne Hansford (dianne@farinhansford.com).

http://dx.doi.org/10.1016/j.cad.2016.08.003
0010-4485/

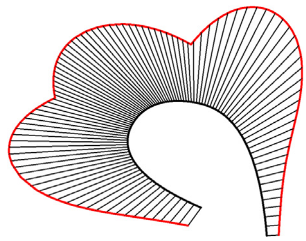

Fig. 1. A curve and its curvature comb. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

We might use arc length s instead of the parameter t. Then, the curvature comb changes appearance because of the resulting change of sampling points on the curve. Using arc length gives a more fine-tuned tool, but it is harder to compute. Plotting comb teeth with arc length spacing typically gives a more even distribution along the curve by eliminating effects of the curve traversal according to the parameter t. Such a traversal change would replace the given parameter t by a transformed parameter $\hat{t}$. If this transformation is only C^0, the comb would not change, but the curvature plot will.

For the comb plots to follow, we render combs as parametric surfaces

$$\mathbf{c}(t,u) = \mathbf{x}(t) - d \cdot u \cdot \kappa(t)\mathbf{n}(t) \tag{2}$$

with t ranging over the curve's domain and $0 \le u \le 1$. This eliminates consideration of how to plot a family of comb teeth.

Combs are encountered as "evolutes" in differential geometry, see [4–6].

In differential geometry, the *Frenet–Serret* equations are given by

$$\mathbf{t}' = -\kappa\mathbf{n}$$
$$\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}$$
$$\mathbf{b}' = -\tau\mathbf{n}$$

where the prime denotes differentiation with respect to arc length and τ is the torsion. See [4,1].

We see that $\mathbf{n}'$ depends on both curvature and torsion, hence does $\mathbf{n}$, and the name "curvature comb" really should be "curvature–torsion comb" for space curves. Here, we simply use "comb".

The combination of curvature and torsion information of the comb may give a designer a more complex tool than just using the curvature plot. We point out here that for true 3D curves, this may be more information than is useful in a typical design situation.

5. Usage

Many commercial CAD systems (such as Alias, Catia, or NX) employ combs for curve shape analysis. A more elementary approach is to utilize the curvature plot. We compare both in Figs. 2 and 3.

The comb plot is the result of superimposing $\mathbf{c}$ over $\mathbf{x}$. Its appearance is thus a combination of two curve shapes, making it harder to judge than the simple curvature plot. While space curves cannot have inflection points, in some views combs may suggest nonexistent inflection points. Thus any interpretation of the 3D comb's shape is view-dependent, hardly making for an objective analysis tool. In addition, the constant d in (1) influences the comb's appearance—a judicial choice is needed for every case. In fact, different parts of the curve might necessitate different values of d.

In many applications, only the shape of planar curves is analyzed. In many others, consideration of 3D curves is necessary. For example, the feature curves on the hood of a car are 3D, and a

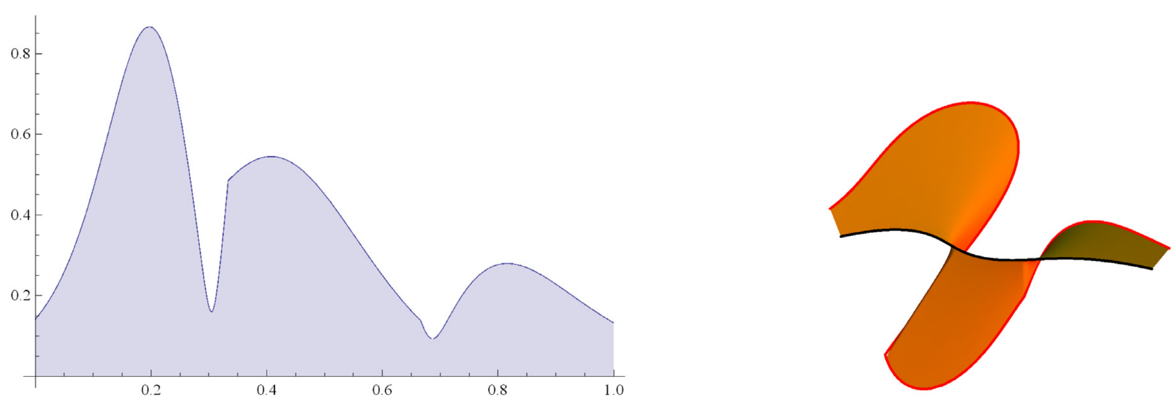

Fig. 2. This particular view of the comb surface (right) suggests two inflection points. Yet none exist, as demonstrated by the curvature plot (left). Indeed, for non-planar curves (like this one), the notion of "inflection point" does not even make sense.

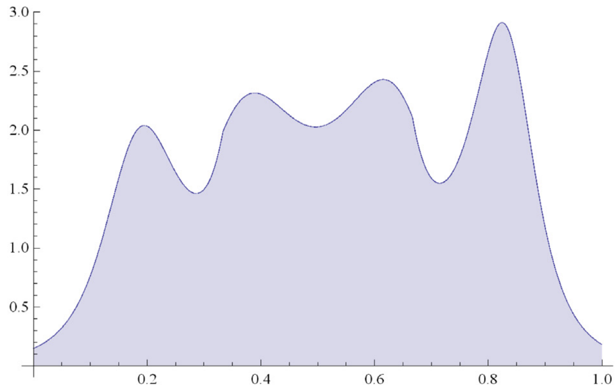

Fig. 3. The uneven curvature behavior is captured by the curvature plot (left), but not by the comb (right).

2D analysis is inappropriate. If we were to design a roller coaster, it is also obvious that planar curves do not enter the picture.

The comb plot allows for a natural correspondence between the comb and the curve. In a curvature plot, points on the curvature graph are disconnected from points on the curve. However, all points with $\kappa'(t) = 0$ can be marked on the curve.

Curvature plots lend themselves to the display of curvature or radius of curvature numerical values. Curvature plots may require additional graphics tools such as extra window environments, which in all fairness will add complexity to the user interface.

One frequently encounters the problem of considering several curves at once. These are typically interrelated (parallel) surface 2D cross sections. The analysis of this whole family of curves is important, and just a set of curvature plots does not give adequate information. More research is needed here.

6. Conclusion

We explored some properties of combs. For 3D curves, they are themselves 3D objects offering a complex mix of information and require a 3D visualization environment. By comparison, curvature plots are simpler and, as a consequence, omit 3D information that might be misleading to the designer.

Based on our studies, we recommend using curvature plots for true space curves and combs for nearly planar curves. Of course, the actual choice should be application-dependent. Curvature plots are best for the initial design of one curve. Combs excel when multiple curves are displayed.

References

[1] Farin G. Curves and Surfaces for Computer Aided Geometric Design. 5th ed. Morgan-Kaufmann; 2001.
[2] Burchardt H, Ayers J, Frey W, Sapidis N. Approximation with aesthetic constraints. In: Sapidis N, editor. Designing Fair Curves and Surfaces. Philadelphia: SIAM; 1994. p. 3–28.
[3] Farin G, Sapidis N. Curvature and the fairness of curves and surfaces. IEEE Comput Graph Appl 1989;9(2):52–7.
[4] do Carmo M. Differential Geometry of Curves and Surfaces. Englewood Cliffs: Prentice Hall; 1976.
[5] Hoschek J. Detecting regions with undesirable curvature. Comput Aided Geom Design 1984;1(2):183–92.
[6] Yates RC. A Handbook on Curves and their Properties. J. W. Edwards; 1952.