# 1

# Affine Maps in 3D

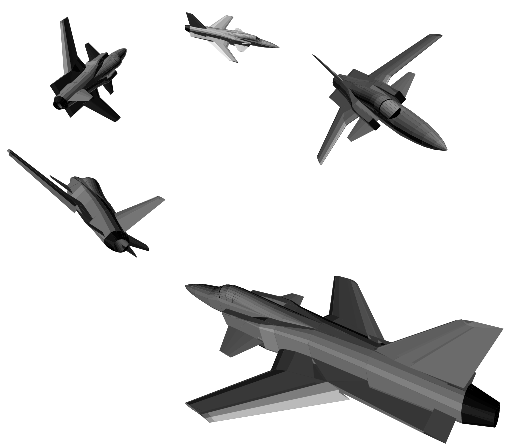


**Figure 1.1.**

Affine maps in 3D: fighter jets twisting and turning through 3D space.

This chapters wraps up the basic geometry tools. Affine maps in 3D are a primary tool for modeling and computer graphics. Figure 1.1 illustrates the use of various affine maps. This chapter goes a little farther than just affine maps by introducing projective maps – the maps used to create realistic 3D images.

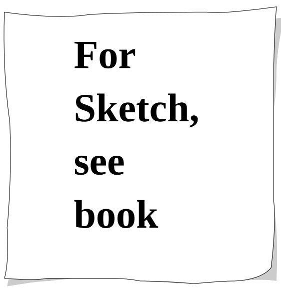
## 1.1 Affine Maps

Linear maps relate vectors to vectors. Affine maps relate points to points. A 3D affine map is written just as a 2D one, namely as

$$\mathbf{x}' = \mathbf{p} + A(\mathbf{x} - \mathbf{o}). \tag{1.1}$$

In general we will assume that the origin of $\mathbf{x}$'s coordinate system has three zero coordinates, and drop the $\mathbf{o}$ term:

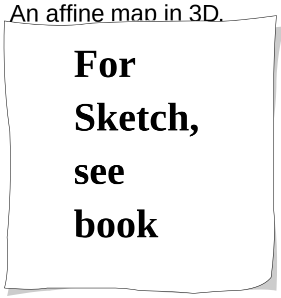
$$\mathbf{x}' = \mathbf{p} + A\mathbf{x}. \tag{1.2}$$




**Sketch 1.**
An affine map in 3D.

Sketch 1.1 gives an example. Recall, the column vectors of $A$ are the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. The point $\mathbf{p}$ tells us where to move the origin of the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system; again, the real action of an affine map is captured by the matrix. Thus by studying matrix actions, or linear maps, we will learn more about affine maps.

We now list some of the important properties of 3D affine maps. They are straightforward generalizations of the 2D cases, and so we just give a brief listing.




**Sketch 2.**
Affine maps leave ratios invari-

1. Affine maps leave *ratios* invariant (see Sketch 1).
2. Affine maps take *parallel planes* to parallel planes (see Figure 1.2).
3. Affine maps take *intersecting planes* to intersecting planes. In particular, the intersection line of the mapped planes is the map of the original intersection line.
4. Affine maps leave *barycentric combinations* invariant. If

$$\mathbf{x} = c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 + c_4\mathbf{p}_4,$$

where $c_1 + c_2 + c_3 + c_4 = 1$, then after an affine map we have

$$\mathbf{x}' = c_1\mathbf{p}_1' + c_2\mathbf{p}_2' + c_3\mathbf{p}_3' + c_4\mathbf{p}_4'.$$

For example, the *centroid* of a tetrahedron will be mapped to the centroid of the mapped tetrahedron (see Sketch 4).


**Sketch 3.**
The centroid is mapped to the centroid.

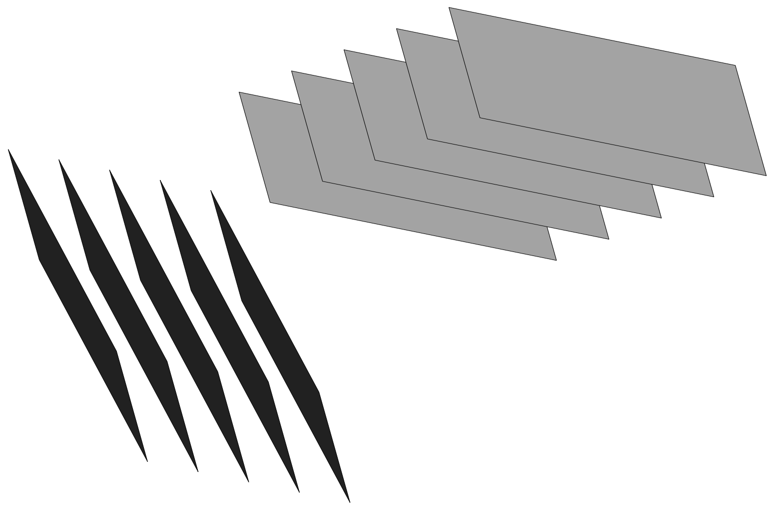


**Figure 1.2.**
Parallel planes get mapped to parallel planes via an affine map.

Most 3D maps do not offer much over their 2D counterparts – but some do. We will go through all of them in detail now.

## 1.2 Translations

A translation is simply (1.2) with $A = I$, the $3 \times 3$ identity matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

that is

$$\mathbf{x}' = \mathbf{p} + I\mathbf{x}.$$

Thus the new $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]-$system has its coordinate axes parallel to the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system. The term $I\mathbf{x} = \mathbf{x}$ needs to be interpreted as a *vector* in the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system for this to make sense! Figure 1.3 shows an example of repeated 3D translations.

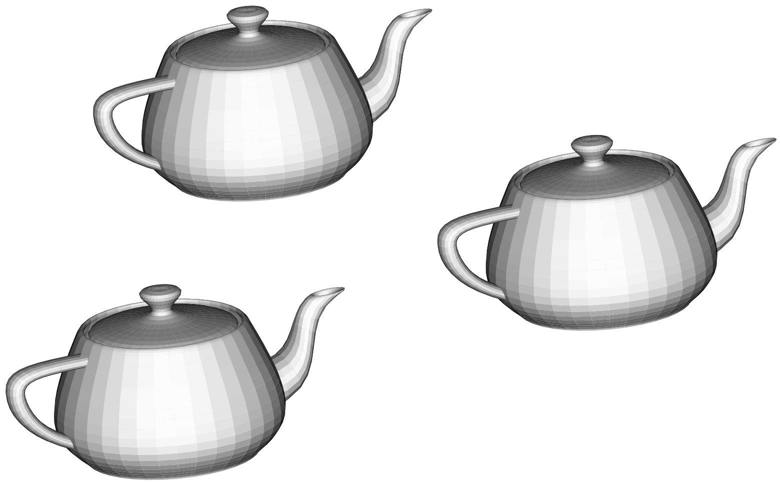


**Figure 1.3.**
Translations in 3D: three translated teapots.

Just as in 2D, a translation is a rigid body motion. The volume of an object is not changed.

## 1.3 Mapping Tetrahedra

A 3D affine map is determined by four point pairs $\mathbf{p}_i \rightarrow \mathbf{p}'_i$ for $i = 1, 2, 3, 4$. In other words, an affine map is determined by a tetrahedron and its image. What is the image of an arbitrary point $\mathbf{x}$ under this affine map?

Affine maps leave *barycentric combinations* unchanged. This will be the key to finding $\mathbf{x}'$, the image of $\mathbf{x}$. If we can write $\mathbf{x}$ in the form

$$\mathbf{x} = u_1\mathbf{p}_1 + u_2\mathbf{p}_2 + u_3\mathbf{p}_3 + u_4\mathbf{p}_4, \tag{1.3}$$

then we know that the image has the same relationship with the $\mathbf{p}'_i$:

$$\mathbf{x}' = u_1\mathbf{p}'_1 + u_2\mathbf{p}'_2 + u_3\mathbf{p}'_3 + u_4\mathbf{p}'_4. \tag{1.4}$$

So all we need to do is find the $u_i$! These are called the *barycentric coordinates* of $\mathbf{x}$ with respect to the $\mathbf{p}_i$, quite in analogy to the triangle case (Section **??**).

We observe that (1.3) is short for three individual coordinate equations. Together with the barycentric combination condition

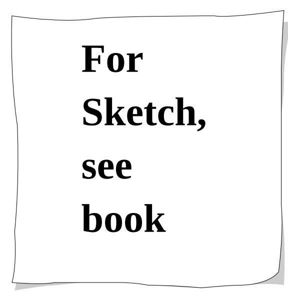
$$u_1 + u_2 + u_3 + u_4 = 1,$$

we have four equations for the four unknowns $u_1, \ldots, u_4$, which we can solve by consulting Chapter **??**.

EXAMPLE 1.1

Let the original tetrahedron be given by the four points $\mathbf{p}_i$

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix}.$$




**Sketch 4.**
An example tetrahedron map.

Let's assume we want to map this tetrahedron to the four points $\mathbf{p}'_i$

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\-1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\-1\end{bmatrix}.$$

This is a pretty straightforward map if you consult Sketch 1.3.

Let's see where the point $\mathbf{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ ends up! First, we find that

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = -2\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix},$$

i.e., the barycentric coordinates of $\mathbf{x}$ with respect to the original $\mathbf{p}_i$ are $(-2, 1, 1, 1)$. Note how they sum to one! Now it is simple to compute the image of $\mathbf{x}$; compute $\mathbf{x}'$ using the same barycentric coordinates with respect to the $\mathbf{p}_i'$:

$$\mathbf{x}' = -2\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}-1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}.$$

A different approach would be to find the $3 \times 3$ matrix $A$ and point $\mathbf{p}$ which describe the affine map. Construct a coordinate system from the $\mathbf{p}_i$ tetrahedron. One way to do this is to choose $\mathbf{p}_1$ as the origin[1] and the three axes are defined as $\mathbf{p}_i - \mathbf{p}_1$ for $i = 2, 3, 4$. The coordinate system of the $\mathbf{p}_i'$ tetrahedron must be based on the same indices. Once we have defined $A$ and $\mathbf{p}$ then we will be able to map $\mathbf{x}$ by this map:

$$\mathbf{x}' = A[\mathbf{x} - \mathbf{p}_1] + \mathbf{p}_1'$$

Thus the point $\mathbf{p} = \mathbf{p}_1'$. In order to determine $A$, let's write down some known relationships. Referring to Sketch 1.3, we know

$$\begin{aligned} A[\mathbf{p}_2 - \mathbf{p}_1] &= \mathbf{p}_2' - \mathbf{p}_1', \\ A[\mathbf{p}_3 - \mathbf{p}_1] &= \mathbf{p}_3' - \mathbf{p}_1', \\ A[\mathbf{p}_4 - \mathbf{p}_1] &= \mathbf{p}_4' - \mathbf{p}_1', \end{aligned}$$

which may be written matrix form as

$$A\begin{bmatrix}\mathbf{p}_2 - \mathbf{p}_1 & \mathbf{p}_3 - \mathbf{p}_1 & \mathbf{p}_4 - \mathbf{p}_1\end{bmatrix} = \begin{bmatrix}\mathbf{p}_2' - \mathbf{p}_1' & \mathbf{p}_3' - \mathbf{p}_1' & \mathbf{p}_4' - \mathbf{p}_1'\end{bmatrix}. \tag{1.5}$$

Thus

$$A = \begin{bmatrix}\mathbf{p}_2' - \mathbf{p}_1' & \mathbf{p}_3' - \mathbf{p}_1' & \mathbf{p}_4' - \mathbf{p}_1'\end{bmatrix}\begin{bmatrix}\mathbf{p}_2 - \mathbf{p}_1 & \mathbf{p}_3 - \mathbf{p}_1 & \mathbf{p}_4 - \mathbf{p}_1\end{bmatrix}^{-1}, \tag{1.6}$$

and $A$ is defined.

**For Sketch, see book**

**Sketch 5.**
The relationship between tetrahedra.

[1] Any of the four $\mathbf{p}_i$ would do, so for the sake of concreteness, we pick the first one.

EXAMPLE 1.2

Revisiting the previous example, we now want to construct the matrix $A$. By selecting $\mathbf{p}_1$ as the origin for the $\mathbf{p}_i$ tetrahedron coordinate system there is no translation; $\mathbf{p}_1$ is the origin in the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system and $\mathbf{p}'_1 = \mathbf{p}_1$. We now compute $A$. ($A$ is the product matrix in the bottom right position):

$$\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{array}$$

In order to compute $\mathbf{x}'$, we have

$$\mathbf{x}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

This is the same result as in the previous example.

## 1.4 Projections

Take any object made of wires outside; let the sun shine on it, and you can observe a shadow. This shadow is the *parallel projection* of your object onto a plane. Everything we draw is a projection of necessity – paper is 2D, after all, whereas most interesting objects are 3D. Figure 1.4 gives an example. Also see Figure **??**.

Projections reduce dimensionality; as basic linear maps, we encountered them in Sections **??** and **??**. As affine maps, they map 3D points onto a plane. In most cases, we are interested in the case of these planes being the coordinate planes. All we have to do then is set one of the point's coordinates to zero.

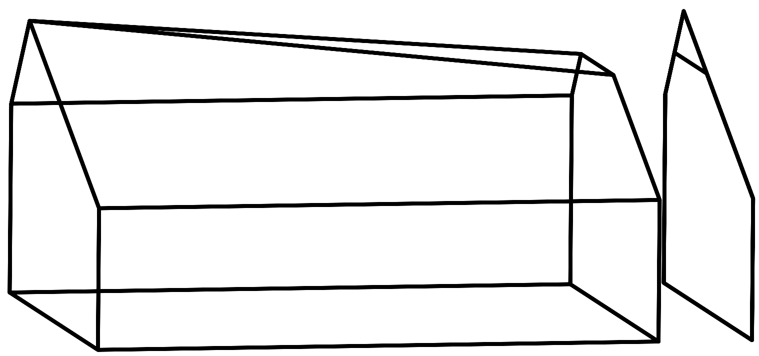


**Figure 1.4.**
Projections: a parallel projection of a 3D barn.


**Sketch 6.**
Projecting a point on a plane.

These basic projections are called *orthographic*: the simulated light ray hits the projection plane at a right angle.

It is a little more interesting to use directions which are at arbitrary angles to the projection plane. These are called *oblique projections*. Let $\mathbf{x}$ be the 3D point to be projected, let $\mathbf{n} \cdot [\mathbf{q} - \mathbf{x}] = 0$ be the projection plane, and let $\mathbf{v}$ indicate the projection direction (see Sketch 1.4).

We have already encountered this problem in Section **??** where it is called line/plane intersection. There, we established that $\mathbf{x}'$, the image of $\mathbf{x}$ under the projection is given by (**??**), which we repeat here:

$$\mathbf{x}' = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v}. \tag{1.7}$$

Figure 1.5 illustrates a simple oblique projection of a cube defined over $[-1, 1]$ in each coordinate with

$$\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

Figure 1.6 creates a projection plane that is not one of the coordinate planes; specifically,

$$\mathbf{n} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$

Finally, Figure 1.7 creates a general oblique projection with

$$\mathbf{n} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

Revisiting the helix, Figure 1.8 is a projection with the same $\mathbf{n}$ and $\mathbf{v}$ as in the previous figure.

How do we write this as an affine map? Without much effort, we find

$$\mathbf{x}' = \mathbf{x} - \frac{\mathbf{n} \cdot \mathbf{x}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v}.$$

We know that we may write dot products in matrix form (see Section **??**):

$$\mathbf{x}' = \mathbf{x} - \frac{\mathbf{n}^{\mathrm{T}}\mathbf{x}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v}.$$

Next, we observe that

$$[\mathbf{n}^{\mathrm{T}} \cdot \mathbf{x}]\mathbf{v} = \mathbf{v}[\mathbf{n}^{\mathrm{T}}\mathbf{x}].$$

Since matrix multiplication is associative (see Section **??**), we also have

$$\mathbf{v}[\mathbf{n}^{\mathrm{T}}\mathbf{x}] = [\mathbf{v}\mathbf{n}^{\mathrm{T}}]\mathbf{x},$$

and thus

$$\mathbf{x}' = [I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v} \cdot \mathbf{n}}]\mathbf{x} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v}. \tag{1.8}$$

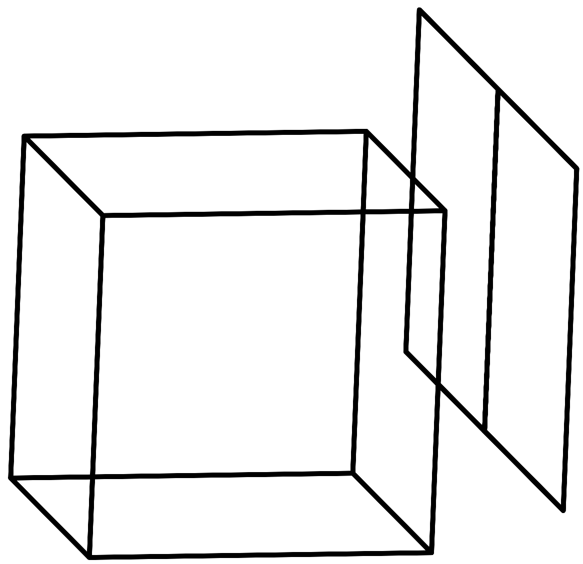


**Figure 1.5.**

Projections: a cube projected in a coordinate plane with an oblique angle.

This is of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{p}$ and hence is an affine map![2]

The term $\mathbf{v}\mathbf{n}^{\mathrm{T}}$ might appear odd, yet it is well-defined. It is a $3 \times 3$ matrix, as in the following example.

EXAMPLE 1.3

| | 1 | 3 | 3 |
|---|---|---|---|
| 1 | 1 | 3 | 3 |
| 2 | 2 | 6 | 6 |
| 0 | 0 | 0 | 0 |

All rows of this matrix are multiples of each other; so are

[2]Technically we should add the origin in order for $\mathbf{p}$ to be a point.

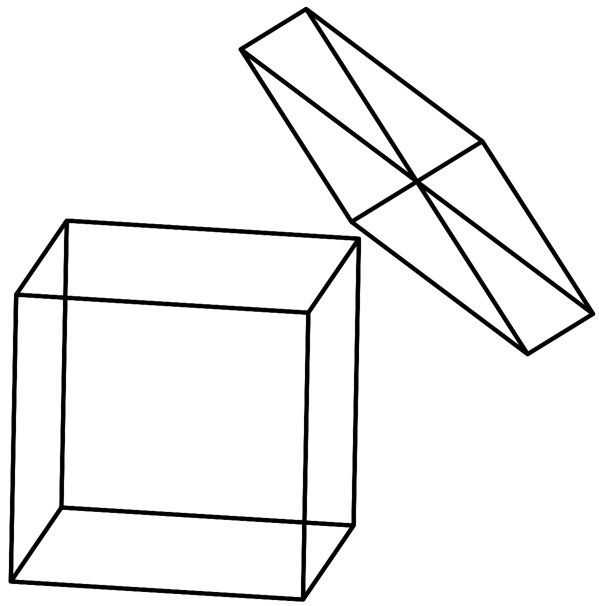


**Figure 1.6.**
Projections: a cube projected in an arbitrary plane with a right angle.

all columns. Matrices which are generated like this are called *dyadic*; their rank is one.

EXAMPLE 1.4

Let a plane be given by $x_1 + x_2 + x_3 - 1 = 0$, a point $\mathbf{x}$ and a direction $\mathbf{v}$ by

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

If we project $\mathbf{x}$ along $\mathbf{v}$ onto the plane, what is $\mathbf{x}'$? First, we need the plane's normalized normal. Calling it $\mathbf{n}$, we have

$$\mathbf{n} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

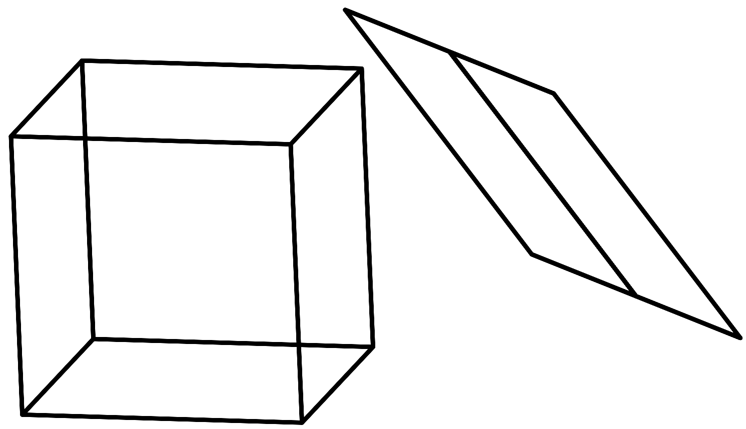


**Figure 1.7.**
Projections: a cube projected in an arbitrary plane with an oblique angle.

Now choose a point $\mathbf{q}$ in the plane. Let's choose $\mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for simplicity. Now we are ready to calculate the quantities in (1.8):

$$\mathbf{v} \cdot \mathbf{n} = -1/\sqrt{3},$$

$$\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{1/\sqrt{3}} =$$

$$\begin{array}{c|ccc} & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{array},$$

$$\frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

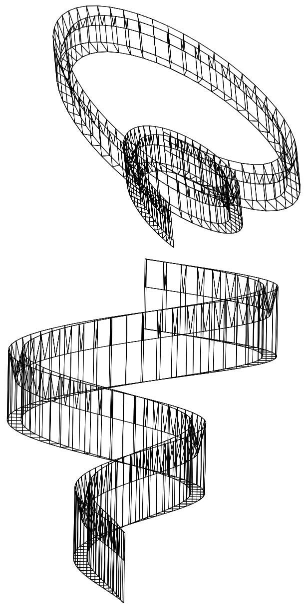


**Figure 1.8.**

Projections: a helix projected in an arbitrary plane with an oblique angle.

Putting all the pieces together:

$$\mathbf{x}' = [I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

Just to double check, enter $\mathbf{x}'$ into the plane equation

$$3 + 3 - 4 - 1 = 0,$$

and we see that

$$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which together verify that this is the correct point.

Sketch 1.4 should convince you that this is indeed the correct answer.

**For Sketch, see book**

**Sketch 7.**

A projection example.

Which of the two possibilities, (1.7) or the affine map (1.8) should you use? Clearly (1.7) is more straightforward and less involved. Yet in some computer graphics or CAD system environments, it may be desirable to have all maps in a unified format, i.e., $A\mathbf{x} + \mathbf{p}$.

## 1.5 Homogeneous Coordinates and Perspective Maps

There is a way to condense the form $\mathbf{x}' = A\mathbf{x} + \mathbf{p}$ of an affine map into just one matrix multiplication

$$\underline{\mathbf{x}}' = M\underline{\mathbf{x}}. \tag{1.9}$$

This is achieved by setting

$$M = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & p_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & p_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\underline{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{x}}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{bmatrix}.$$

The 4D point $\underline{\mathbf{x}}$ is called the *homogeneous form* of the affine point $\mathbf{x}$. You should verify for yourself that (1.9) is indeed the same affine map as before!

The homogeneous form is more general than just adding a fourth coordinate $x_4 = 1$ to a point. If, perhaps as the result of some computation, the fourth coordinate does not equal one, one gets from the homogeneous point $\underline{\mathbf{x}}$ to its affine counterpart $\mathbf{x}$ by dividing through by $x_4$. Thus one affine point has infinitely many homogeneous representations!

EXAMPLE 1.5

(The symbol $\approx$ should be read "corresponds to".)

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \approx \begin{bmatrix} 10 \\ -10 \\ 30 \\ 10 \end{bmatrix} \approx \begin{bmatrix} -2 \\ 2 \\ -6 \\ -2 \end{bmatrix}.$$

This example shows two homogeneous representations of one affine point.

Using the homogeneous matrix form of (1.9), the matrix $M$ for the point into a plane projection from (1.8) becomes

$$\left[\begin{array}{c|c} \begin{bmatrix} \mathbf{v}\cdot\mathbf{n} & 0 & 0 \\ 0 & \mathbf{v}\cdot\mathbf{n} & 0 \\ 0 & 0 & \mathbf{v}\cdot\mathbf{n} \end{bmatrix} - \mathbf{v}\mathbf{n}^{\mathrm{T}} & (\mathbf{q}\cdot\mathbf{n})\mathbf{v} \\ \hline 0 \quad 0 \quad 0 & \mathbf{v}\cdot\mathbf{n} \end{array}\right].$$

Here, the element $m_{4,4} = \mathbf{v}\cdot\mathbf{n}$. Thus $\underline{x}_4 = \mathbf{v}\cdot\mathbf{n}$, and we will have to divide $\underline{\mathbf{x}}$'s coordinates by $\underline{x}_4$ in order to obtain the corresponding affine point.

A simple change in our equations will lead us from parallel projections onto a plane to *perspective projections*. Instead of using a constant direction $\mathbf{v}$ for all projections, now the direction depends on the point $\mathbf{x}$. More precisely, let it be the line from $\mathbf{x}$ to the origin of our coordinate system. Then, as shown in Sketch 1.5, $\mathbf{v} = -\mathbf{x}$, and (1.7) becomes

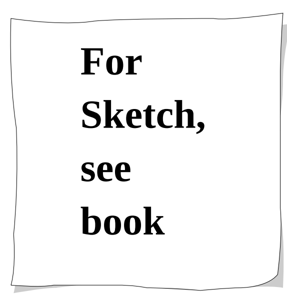
$$\mathbf{x}' = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}]\cdot\mathbf{n}}{\mathbf{x}\cdot\mathbf{n}}\mathbf{x},$$

which quickly simplifies to

$$\mathbf{x}' = \frac{\mathbf{q}\cdot\mathbf{n}}{\mathbf{x}\cdot\mathbf{n}}\mathbf{x}. \tag{1.10}$$

In homogeneous form, this is described by the following matrix

$$M: \left[\begin{array}{ccc|c} \multicolumn{3}{l|}{I[\mathbf{q}\cdot\mathbf{n}]} & \mathbf{o} \\ \hline 0 & 0 & 0 & \mathbf{x}\cdot\mathbf{n} \end{array}\right].$$




**Sketch 8.**
Perspective projection.

Perspective projections are not affine maps anymore! To see this, a simple example will suffice.

EXAMPLE 1.6

Take the plane $x_3 = 1$; let $\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ be a point on the plane. Now $\mathbf{q} \cdot \mathbf{n} = 1$ and $\mathbf{x} \cdot \mathbf{n} = x_3$, resulting in the map

$$\mathbf{x}' = \frac{1}{x_3}\mathbf{x}.$$

Take the three points

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}.$$

This example is illustrated in Sketch 1.5. Note that $\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_3$, i.e., $\mathbf{x}_2$ is the midpoint of $\mathbf{x}_1$ and $\mathbf{x}_3$.

Their images are

$$\mathbf{x}_1' = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2' = \begin{bmatrix} 1 \\ -1/3 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3' = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

The perspective map destroyed the midpoint relation! For now $\mathbf{x}_2' = \frac{2}{3}\mathbf{x}_1' + \frac{1}{3}\mathbf{x}_3'$.

Thus the ratio of three points is changed by perspective maps. As a consequence, two parallel lines will not be mapped to parallel lines. Because of this effect, perspective maps are a good model for how we perceive 3D space around us. Parallel lines do seemingly intersect in a distance, and are thus not *perceived* as being parallel! Figure 1.9 is a parallel projection and Figure 1.10 illustrates the same geometry with a perspective projection. Notice in the perspective image, the sides of the bounding cube that move into the page are no longer parallel.

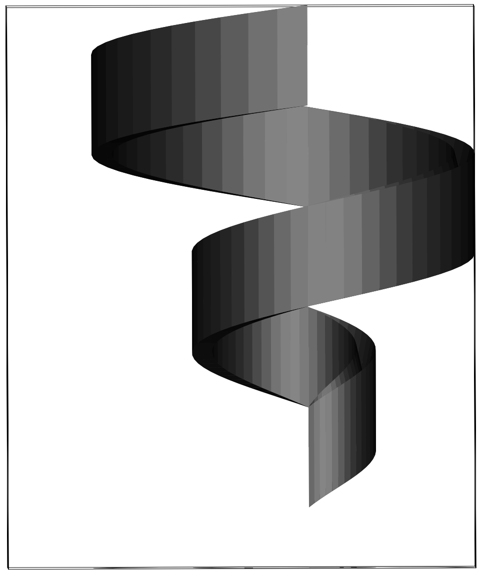


**Figure 1.9.**

Parallel projection: A 3D helix and two orthographic projections on the left and bottom walls of the bounding cube – not visible due to the orthographic projection used for the whole scene.

The study of perspective goes back to the fourteenth century – before that, artists simply could not draw realistic 3D images. One of the foremost researchers in the area of perspective maps was A. Dürer; see Figure 1.11 for one of his experiments.[3]

## 1.6 Exercises

We'll use four points

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

[3]From *The Complete Woodcuts of Albrecht Dürer*, edited by W. Durth, Dover Publications Inc., New York, 1963. A website with material on Dürer: `http://www.bilkent.edu.tr/wm/paint/auth/durer/`.

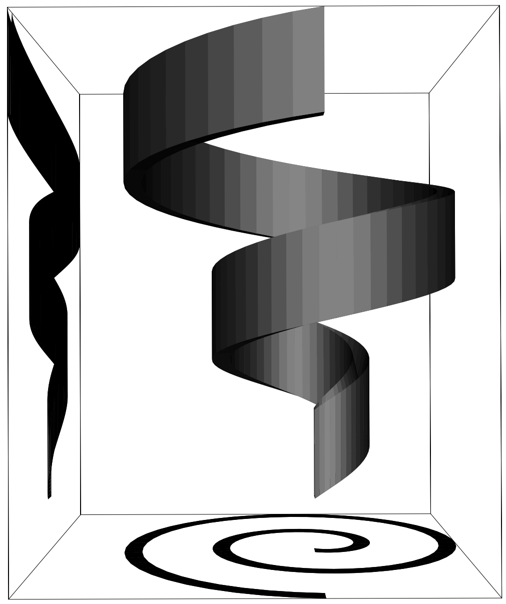


**Figure 1.10.**

Perspective projection: A 3D helix and two orthographic projections on the left and bottom walls of the bounding cube – visible due to the perspective projection used for the whole scene.

and four points

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and also the plane through $\mathbf{q}$ with normal $\mathbf{n}$:

$$\mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{n} = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}.$$

1. Using a direction

$$\mathbf{v} = \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix},$$

**Figure 1.11.**

Perspective maps: an experiment by A. Dürer.

what are the images of the $\mathbf{x}_i$ when projected onto the plane with this direction?

2. Using the same $\mathbf{v}$ as in the previous problem, what are the images of the $\mathbf{y}_i$?

3. What are the images of the $\mathbf{x}_i$ when projected onto the plane by a perspective projection through the origin?

4. What are the images of the $\mathbf{y}_i$ when projected onto the plane by a perspective projection through the origin?

5. Compute the centroid $\mathbf{c}$ of the $\mathbf{x}_i$ and then the centroid $\mathbf{c}'$ of their perspective images (previous Exercise). Is $\mathbf{c}'$ the image of $\mathbf{c}$ under the perspective map?

6. An affine map $\mathbf{x}_i \rightarrow \mathbf{y}_i; i = 1, 2, 3, 4$ is uniquely defined. What is it?

7. What is the image of

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

under the map from the previous problem? Use two ways to compute it.

8. What are the geometric properties of the affine map from the last two problems?

9. We claimed that (1.8) reduces to (1.10). This necessitates that

$$[I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{n} \cdot \mathbf{v}}]\mathbf{x} = \mathbf{0}.$$

Show that this is indeed true.